

SPECTRAL FLOW IN FREDHOLM MODULES, ETA INVARIANTS AND THE JLO COCYCLE ¹

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ABSTRACT. We give a comprehensive account of an analytic approach to spectral flow along paths of self-adjoint Breuer-Fredholm operators in a type I_∞ or II_∞ von Neumann algebra \mathcal{N} . The framework is that of *odd unbounded θ -summable Breuer-Fredholm modules* for a unital Banach $*$ -algebra, \mathcal{A} . In the type II_∞ case spectral flow is real-valued, has no topological definition as an intersection number and our formulae encompass all that is known. We borrow Ezra Getzler's idea (suggested by I. M. Singer) of considering spectral flow (and eta invariants) as the integral of a closed one-form on an affine space. Applications in both the type I and type II cases include a general formula for the relative index of two projections, representing truncated eta functions as integrals of one forms and expressing spectral flow in terms of the JLO cocycle to give the pairing of the K -homology and K -theory of \mathcal{A} .

1. INTRODUCTION

The notion of spectral flow has been a useful analytic tool in geometry ever since its invention by Atiyah and Lusztig [APS1, APS3, BW]. Motivated by observations of I.M. Singer [Si] on eta invariants which suggest that spectral flow should be expressible as the integral of a one-form, there has been a succession of contributions in [DHK],[H],[Kam],[G],[Ph, Ph1] and [P1, P2]. A key step in synthesising these developments was taken in [CP1] where we exploited previous work by one of us [Ph, Ph1] and ideas of [G] to produce spectral flow formulae as integrals of one-forms on affine spaces arising from finitely summable Fredholm modules. Significantly, we also established spectral flow formulae along paths of self-adjoint Breuer-Fredholm operators in a type II_∞ von Neumann algebra, a development that was first hinted at in [APS3]. This has special relevance to recent developments in the study of L^2 spectral invariants for manifolds whose fundamental group has a non-type I regular representation, see [Ma] for a review of these ideas. The casual reader of say [G] might be puzzled as to the reasons for our lengthy treatment. They are threefold, first of all we establish much more general formulae than are described in [G]. Second, in the type II_∞ case there is no topological definition of spectral flow as an intersection number (partly due to the fact that Breuer-Fredholm operators may have zero in their continuous spectrum). Third, the use of the Duhamel Principle in [G] requires strong domain assumptions (which are not stated) for the unbounded operators that appear there. Inserting these assumptions then excludes interesting examples and introduces ugly unnecessary complications. We adopt new methods which entail analytic subtleties in both the type I_∞ and type II_∞ case but lead to analytic results of wider interest and applicability and considerably extend the existing literature even in the classical type I case (for example, our formula for the relative index of two projections in Section 3).

The present paper also goes much further in explaining these earlier developments in terms of a single general formula for spectral flow along paths of self-adjoint Breuer-Fredholm operators in θ -summable Breuer-Fredholm modules. Moreover our proofs are the same for both type I_∞ and type II_∞ von Neumann algebras. Our analytic approach has already found application in the type I_∞ case in [BF] and in proving index theorems for generalised Toeplitz operators [CPSu]. In this paper our main application is on the relationship with

the JLO cocycle (and hence the pairing of the K -homology of \mathcal{A} and the odd K -theory of \mathcal{A}).

To explain our results we need some notation and definitions. We denote by \mathcal{N} a type I_∞ or type II_∞ von Neumann algebra acting on a Hilbert space H with faithful normal semifinite trace τ . The ideal of “compact operators” in \mathcal{N} is denoted $\mathcal{K}_\mathcal{N}$ (see [B1, B2] for this and also the notion of Fredholm operator in the type II_∞ setting). We fix an unbounded self-adjoint operator D_0 on H affiliated with \mathcal{N} and assume that A is a self-adjoint element in \mathcal{N} (the set of all such being denoted \mathcal{N}_{sa}). We consider paths of the form $D(t) = D_0 + A(t)$ where t is a real parameter and $A(t) \in \mathcal{N}_{sa}$ for each t .

Definition. We say that (\mathcal{N}, D_0) is an **unbounded θ -summable Breuer-Fredholm module** for a Banach $*$ -algebra \mathcal{A} if \mathcal{A} is represented in \mathcal{N} and if $e^{-tD_0^2}$ is trace-class for all $t > 0$ and $[D_0, a]$ is bounded for all a in a dense $*$ -subalgebra of \mathcal{A} .

We let

$$\mathcal{M}_0 = \{D = D_0 + A \mid A \in \mathcal{N}_{sa}\}$$

Clearly \mathcal{M}_0 is an affine space modelled on \mathcal{N}_{sa} . Introduce the ‘gauge group’ \mathcal{G} defined by

$$\mathcal{G} = \{U \in \mathcal{N} \mid U \text{ is unitary, } [D_0, U] \text{ is bounded}\}.$$

Let $\gamma = \{D_t = D_0 + A(t), a \leq t \leq b\}$ be a piecewise C^1 path in \mathcal{M}_0 with D_a and D_b invertible. The spectral flow formula of [G] when $\mathcal{N} = B(H)$ is

$$\begin{aligned} \text{sf}(D_a, D_b) &= - \int_\gamma \alpha_\epsilon + \frac{1}{2} \eta_\epsilon(D_b) - \frac{1}{2} \eta_\epsilon(D_a) \\ &= \sqrt{\frac{\epsilon}{\pi}} \int_a^b \tau \left(\frac{d}{dt} (D_t) e^{-\epsilon D_t^2} \right) dt + \frac{1}{2} \eta_\epsilon(D_b) - \frac{1}{2} \eta_\epsilon(D_a). \end{aligned}$$

where $\eta_\epsilon(D)$ are approximate eta invariant correction terms (we define these later). The difference vanishes if the endpoints are in the same gauge group orbit.

Our aim is to establish not only this formula but a much more general one for the spectral flow of bounded self-adjoint Breuer-Fredholm operators. We will use this to establish formulae for spectral flow along paths in \mathcal{M}_0 . To describe the bounded case we need some further notation and definitions.

Definition. A bounded, **odd Breuer-Fredholm module** for a unital Banach $*$ -algebra \mathcal{A} represented in \mathcal{N} is a pair (\mathcal{N}, F) with F a self-adjoint operator in \mathcal{N} such that $F^2 = 1$ and $[F, a] \in \mathcal{K}_\mathcal{N}$ for all $a \in \mathcal{A}$.

In this introduction let us restrict to the type I_∞ case and introduce the two-sided ideal of operators $Li_0(H)$ consisting of those compact operators T whose n^{th} singular value is $o((\log n)^{-1})$ as $n \rightarrow \infty$. Then we say (H, F) is θ -summable if $[F, a] \in Li_0^{1/2}(H)$ for a dense set of $a \in \mathcal{A}$. The choice $F = \text{sign}(D_0)$ relates unbounded to bounded Fredholm modules. If we replace the ideal $Li_0(H)$ by the ideal $Li(H)$ consisting of compact operators T whose n^{th} singular value is $O((\log n)^{-1})$, then this defines a weakly θ -summable Fredholm module. (Connes’ most recent definition of θ -summable is what we call weakly θ -summable [Co4].) We prove similar spectral flow formulae in each of these cases.

♠ Our most general formula in the bounded case deals with a pair of self-adjoint Fredholm operators $\{F_j, \quad j = 1, 2\}$, joined by a piecewise C^1 path $\{F_t\}$, $t \in [1, 2]$ in a certain affine subspace (specified in terms of $Li_0(H)$) of the space of all self-adjoint Fredholm operators. The spectral flow along such a path is given by

$$sf(F_1, F_2) = \frac{1}{C} \int_1^2 \tau \left(\frac{d}{dt}(F_t) |1 - F_t^2|^{-r} e^{-|1 - F_t^2|^{-\sigma}} \right) dt + \gamma(F_2) - \gamma(F_1)$$

where the $\gamma(F_j)$ are eta invariant type correction terms and C is a normalization constant depending on the parameters $r \geq 0$ and $\sigma \geq 1$. To see one important place where such complicated formulae arise, one takes the Getzler expression with $\epsilon = 1$:

$$\frac{1}{\sqrt{\pi}} \int_a^b \tau \left(\frac{d}{dt}(D_t) e^{-D_t^2} \right) dt$$

and does the change of variable $F_t = D_t(1 + D_t^2)^{-1/2}$. Then, $(1 + D_t^2)^{-1} = (1 - F_t^2) = |1 - F_t^2|$, and if one is careless and just differentiates formally (not worrying about the order of the factors), one obtains the expression:

$$\frac{e}{\sqrt{\pi}} \int_a^b \tau \left(\frac{d}{dt}(F_t) |1 - F_t^2|^{-3/2} e^{|1 - F_t^2|^{-1}} \right) dt.$$

While the actual details are much more complicated, this is the heuristic essence of our reduction of the unbounded case to the bounded case: see Propositions 6.5 and 6.6. ♠

With the extra flexibility afforded by the parameters $r \geq 0$ and $\sigma \geq 1$, we also show that this same formula, given a type II_∞ analogue of the ideal $Li(H)$, holds for spectral flow along a path in an affine subspace of bounded self-adjoint Breuer-Fredholm operators associated with a θ -summable Breuer-Fredholm module.

Our approach is very general; formulae studied elsewhere follow from it (with the proviso that side conditions are needed in some cases).

The plan of the paper is to relegate many technical functional analytic issues to appendices. This is not to say these results are not in themselves of interest, rather that they could detract from the flow of the main arguments.

We begin in Section 2 by laying out all of our definitions and assumptions and where appropriate indicating how they relate to the existing literature.

Section 3 contains a formula for the essential codimension (or relative index) of two projections (Theorem 3.1). Such results have a long history and we believe our formula is the most general possible. The relevance of this to the notion of spectral flow in Fredholm modules is the following. Given a (bounded) Fredholm module (H, F_0) for a Banach $*$ -algebra \mathcal{A} and a unitary $u \in \mathcal{A}$, then $P = \frac{1}{2}(F_0 + 1)$ and $Q = uPu^*$ are two projections with the property that the operator $QP : P(H) \rightarrow Q(H)$ is a Fredholm operator whose (relative) index equals the spectral flow of the straight line path from F_0 to uF_0u^* . Depending on the summability

flavour of the Fredholm module we are able to obtain explicit integral formulae for this index in Section 4 based on these results of Section 3.

Thus, Section 4 contains the first of our spectral flow formulae (eg., Theorems 4.1 and 4.2). We single these out because they are elegant, their proofs are relatively short and ultimately the proofs of all of our later formulae are based on them.

In Section 5 we show that the integral formulae of Section 4 make sense in the more general context of pre-Fredholm modules where $1 - F_0^2$ is not 0 but merely compact of some summability flavour (these arise naturally from transforming **unbounded** Fredholm modules). We show that the integrals involved are actually integrals of exact one-forms on appropriate affine spaces of the form: $F_0 + \mathcal{I}_{F_0}$ where \mathcal{I}_{F_0} is a certain subspace of compact operators depending on the the summability flavour of the module. The exactness of these one-forms is a difficult analytic problem involving Cauchy integrals along unbounded contours. This problem is a key difficult step in our approach, but in the text we focus on the ideas, relegating most of the technical issues to appendix C. For arbitrary piecewise C^1 paths in the space $F_0 + \mathcal{I}_{F_0}$ from say F_a to F_b we must tack on one extra path at each end so that our new path runs between the corresponding symmetries $\tilde{F}_a = \text{sign}(F_a)$ to $\tilde{F}_b = \text{sign}(F_b)$. We then invoke the invariance of the integral of an exact one-form and the results of Section 4 to obtain our formulae (eg., Theorems 5.7 and 5.9). Thus the correction terms arise naturally as the integrals of the one-forms from an operator F in $F_0 + \mathcal{I}_{F_0}$ to its associated symmetry \tilde{F} .

In Section 6 we show that the transformation from unbounded θ -summable modules (H, D) to bounded modules (H, F) via the map $F_D = D(1 + D^2)^{-1}$ carries C^1 paths to C^1 paths provided we consider θ_q -summable modules in the bounded case (where $0 < q < 1$). The integral formulae are shown to transform as expected, thanks to the commutativity property of the trace (Proposition 6.6).

In Section 7 we then use this transformation to obtain our formulae in the unbounded case from those in the bounded case (Section 5). The direct translations of these formulae involve the parameter q : we obtain our final versions by taking the limit as $q \rightarrow 1$ (Theorem 7.8 and Corollary 7.10).

In Section 8 we shed further light on the idea that the eta invariant is the integral of a one-form. We see that the correction terms in our spectral flow formula correspond to integrating our one-form along a particular path. These correction terms are seen to be truncated eta invariants by using path independence of the integral of our exact one form and integrating along a different path joining the same endpoints (with a discontinuity at one end). In the type I_∞ setting considered by [G] where he assumed that the endpoints of his path, D_a and D_b are invertible, we show that our correction terms are identical with his. In the general case (Theorem 8.9 and Corollary 8.10) we show how to modify the eta terms to remove the invertibility assumption.

In Section 9 we use the Laplace Transform and the results of Section 7 to give a “best possible” finitely summable unbounded version of the spectral flow formula (Theorem 9.3). This should be compared with Theorem 2.16 of [CP1]. In a later paper [CPSu], this result

is crucial in proving Connes' Dixmier-trace formula for the odd index in the general setting of $(1, \infty)$ Breuer-Fredholm modules.

In Section 10 we explain how our formulae lead to the JLO cocycle by generalising the main theorem of [G]. Our formula for type II spectral flow in terms of the JLO cocycle may be used as the starting point for a proof of the local index formula of Connes and Moscovici [CoMo] in the setting of Breuer-Fredholm modules. This is a lengthy matter however and we leave it to another place.

We reiterate that our aim has been to create a usable theory of spectral flow in a type II_∞ von Neumann algebra. In a separate paper [CPSu] we show that our formulae imply index theorems such as those of [CDSS] bringing them into the fold of noncommutative geometry.

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2. DEFINITIONS AND NOTATIONS

Throughout this paper, \mathcal{N} will denote a semifinite von Neumann algebra (with separable predual) and τ will denote a faithful, normal semifinite trace on \mathcal{N} . The norm-closed two-sided ideal in \mathcal{N} generated by the elements of finite trace, will be denoted by $\mathcal{K}_{\mathcal{N}}$. We will be concerned with certain normed ideals \mathcal{I} contained in $\mathcal{K}_{\mathcal{N}}$ and which are best defined in terms of *generalized singular values* a notion due to Fack and Kosaki (and others), see [FK].

2.1. Singular Values.

Definition 2.1. *If $S \in \mathcal{N}$ and $t > 0$, then the **t-th (generalized) singular value** of S is given by*

$$\mu_t(S) = \inf\{\|SE\| \mid E \text{ is a projection in } \mathcal{N} \text{ with } \tau(1 - E) \leq t\}.$$

Although we refer to [FK] for the properties of these singular values, we note that we are restricting ourselves only to bounded operators (in \mathcal{N}). Hence, we have that $0 \leq \mu_t(S) \leq \|S\|$ for all $t > 0$ and that for $S \in \mathcal{K}_{\mathcal{N}}$ we have $\mu_t(S) \rightarrow 0$. Thus, for us it is reasonable to set $\mu_0 = \|S\|$.

2.2. Operator Ideals. ♠ For most of the paper we would like to concern ourselves with the ideals, Li and Li_0 defined below. However, when we transform from the setting of unbounded modules to bounded modules we lose a little control and are forced to consider powers of these ideals, Li^q and Li_0^q for $0 < q \leq 1$. In the end, we are able to rid ourselves of these irritating exponents in our formulae by taking a limit as $q \rightarrow 1$. We observe that $Li_0 \subset Li \subset Li_0^q \subset Li^q$ for all q with $0 < q < 1$. ♠

Definition 2.2. *We define $Li = \{T \in \mathcal{N} \mid \mu_t(T) = O(1/\log t)\}$. The norm on this ideal is*

$$\|T\|_{Li} = \sup_{x>0} \left\{ \frac{\int_0^x \mu_t(T) dt}{\int_0^x (\log(t+e))^{-1} dt} \right\}.$$

We note that $\|T\|_{Li} \geq \|T\|$.

The ideal Li_0 is the closed subspace of Li in the norm $\|\cdot\|_{Li}$ of those operators $T \in \mathcal{N}$ satisfying $\mu_t(T) = o(1/\log t)$.

For $0 < q \leq 1$, we consider also the powers of these ideals $Li^q \supset Li_0^q$ with the norm

$$\|T\|_{Li^q} = (\| |T|^{1/q} \|_{Li})^q.$$

For more on these ideals we refer the reader to appendix A.

We also consider the ideals of finitely summable operators, L^p , which we define for $p \geq 1$ by $L^p = \{T \in \mathcal{N} \mid \tau(|T|^p) < \infty\}$. Since the trace of a positive operator is expressible in terms of singular values [FK], we could also define these ideals in terms of singular values via:

$$L^p = \{T \in \mathcal{N} \mid \int_0^\infty \mu_t(T)^p dt < \infty\}.$$

The norm on L^p is

$$\|T\|_p = \max\{\|T\|, (\tau(|T|^p))^{1/p}\}.$$

We observe that if \mathcal{N} is type I then we can omit the operator norm on the right hand side given the usual normalization of τ . In the type II case, our ideal L^p is strictly contained in the space of p -summable measurable operators [FK].

2.3. Breuer-Fredholm Modules.

Definition 2.3. An odd pre-Breuer-Fredholm module for a unital Banach $*$ -algebra \mathcal{A} is a pair (\mathcal{N}, F_0) where \mathcal{A} is (continuously) represented in \mathcal{N} and F_0 is a self-adjoint Breuer-Fredholm operator in \mathcal{N} satisfying:

1. $1 - F_0^2 \in \mathcal{K}_{\mathcal{N}}$, and
2. $[F_0, a] \in \mathcal{K}_{\mathcal{N}}$ for $a \in \mathcal{A}$.

If $1 - F_0^2 = 0$ we drop the prefix "pre-".

If, in addition, our module satisfies:

- 1.' $1 - F_0^2 \in Li_0$ (respectively, Li ; Li_0^q for $0 < q \leq 1$) and
- 2.' $[F_0, a] \in Li_0^{1/2}$ (respectively, $Li^{1/2}$; $Li_0^{q/2}$) for a dense set of $a \in \mathcal{A}$,

then we call (\mathcal{N}, F_0) **θ -summable** (respectively, **weakly θ -summable**; **θ_q -summable**). By the Remark $\spadesuit \cdots \spadesuit$ above, θ -summable implies weakly θ -summable implies θ_q -summable for $0 < q < 1$. We note that θ_1 -summable = θ -summable.

We warn the reader that what we call weakly θ -summable for a Fredholm module is what Connes calls θ -summable in [Co4] chapter IV. We do this to be consistent with Connes' original definitions in the unbounded case discussed below [Co2], [Co3]. We now define the closely related notion of unbounded Breuer-Fredholm modules and note that in the above definition we do not bother with the extra adjective *bounded*.

Definition 2.4. An **odd unbounded Breuer-Fredholm module** for a unital Banach $*$ -algebra \mathcal{A} is a pair (\mathcal{N}, D_0) where \mathcal{A} is (continuously) represented in \mathcal{N} and D_0 is an unbounded self-adjoint operator affiliated with \mathcal{N} satisfying:

1. $(1 + D_0^2)^{-1} \in \mathcal{K}_{\mathcal{N}}$, and
2. $[D_0, a] \in \mathcal{N}$ for a dense set of $a \in \mathcal{A}$.

If, in addition, our module satisfies:

- 1.' $(1 + D_0^2)^{-1} \in Li_0$ (respectively, Li ; Li_0^q for $0 < q \leq 1$),

then we call (\mathcal{N}, D_0) **θ -summable** (respectively, **weakly θ -summable**; **θ_q -summable**). Again, θ -summable \Rightarrow weakly θ -summable $\Rightarrow \theta_q$ -summable for $0 < q < 1$. We also note that θ_1 -summable = θ -summable.

By Corollary B.6 of Appendix B, we observe that for unbounded Fredholm modules our definition of θ -summable agrees with Connes' original definition, [Co2], [Co3], while our definition of weakly θ -summable coincides with his later definition of θ -summable, [Co4].

2.4. The Transformation $D \mapsto D(1 + D^2)^{-1/2}$.

Remarks. In general, if (\mathcal{N}, D_0) is an odd unbounded Breuer-Fredholm module for some \mathcal{A} and $F_0 = D_0(1 + D_0^2)^{-1/2}$, then (\mathcal{N}, F_0) is an odd pre-Breuer-Fredholm module. Since $1 - F_0^2 = (1 + D_0^2)^{-1}$ the conditions labelled 1. in the definitions coincide. The commutator conditions are more subtle, but are handled by the strong-operator convergent integral:

$$F_0 = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} D_0 (1 + D_0^2 + \lambda)^{-1} d\lambda,$$

[BJ], [CP1].

Using this integral formula, we showed in [CP1] that an odd unbounded p -summable Breuer-Fredholm module yields an odd $(p + \epsilon)$ -summable *pre*-Breuer-Fredholm module for each $\epsilon > 0$. By using completely different techniques, F.A. Sukochev [Suk], and then [CPS] were able to eliminate the ϵ (*except* in the case $p = 1$ and \mathcal{N} is type II_∞). Unfortunately, these new techniques do not allow us to handle certain smoothness properties of the transformation $D \mapsto D(1 + D^2)^{-1/2}$ which are crucial in obtaining integral formulae for spectral flow.

♠ In the present paper, we revisit the integral formula (specifically, lemma 2.7 of [CP1]) to show that if the unbounded module (\mathcal{N}, D_0) is θ -summable (even weakly θ -summable) then the **bounded** module (\mathcal{N}, F_0) is θ_q -summable for all q , $0 < q < 1$. If we let $1/q = 1 + \epsilon$ then this is the same ϵ -problem we encountered in the finitely-summable situation (an artifact of lemma 2.7 of [CP1]). In a recent preprint [Suk2], F. A. Sukochev has extended his techniques to the θ -summable case to show that if (\mathcal{N}, D_0) is θ -summable, then so is (\mathcal{N}, F_0) . However, we still need these integral techniques to handle the smoothness of the transformation $D \mapsto D(1 + D^2)^{-1/2}$, and so we are **forced** to consider θ_q -summable modules for $0 < q < 1$. ♠

As in section 1 of [CP1] we can obtain a genuine Breuer-Fredholm module $(\mathcal{N}, \tilde{F}_0)$ from a pre-Breuer-Fredholm module, (\mathcal{N}, F_0) by letting $\tilde{F}_0 = \text{sign}(F_0)$ where

$$\text{sign}(x) = \begin{cases} +1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

We observe that $(\mathcal{N}, \tilde{F}_0)$ has the same summability flavour as (\mathcal{N}, F_0) since

$$(\tilde{F}_0 - F_0) = (1 - F_0^2)(\tilde{F}_0 + F_0)^{-1}.$$

We warn the reader that our sign function is never 0 and so $\text{sign}(F_0)$ is always a self-adjoint unitary: this differs from Connes' convention [Co4].

2.5. Spectral Flow.

Definition 2.5. *If $\{F_t\}$ is a continuous path of self-adjoint Breuer-Fredholm operators in \mathcal{N} , then the definition of the **spectral flow** of the path, $sf(\{F_t\})$ is based on the following sequence of observations in [Ph1]:*

1. *The map $t \mapsto \text{sign}(F_t)$ is usually discontinuous as is the projection-valued mapping $t \mapsto P_t = \frac{1}{2}(\text{sign}(F_t) + 1)$.*

2. *However, if $\pi : \mathcal{N} \rightarrow \mathcal{N}/\mathcal{K}_{\mathcal{N}}$ is the canonical mapping, then $t \mapsto \pi(P_t)$ is continuous.*

3. *If P and Q are projections in \mathcal{N} and $\|\pi(P) - \pi(Q)\| < 1$ then*

$$PQ : \text{rng}(Q) \rightarrow \text{rng}(P)$$

is a Breuer-Fredholm operator and so $\text{ind}(PQ) \in \mathbb{R}$ is well-defined.

4. *If we partition the parameter interval of $\{F_t\}$ so that the $\pi(P_t)$ do not vary much in norm on each subinterval of the partition then*

$$sf(\{F_t\}) := \sum_{i=1}^n \text{ind}(P_{t_{i-1}} P_{t_i})$$

is a well-defined and (path-) homotopy-invariant number which agrees with the usual notion of spectral flow in the type I_{∞} case [Ph], and also agrees with all previous special definitions of type II_{∞} spectral flow, for example [P1, P2].

In particular, if the path, $\{F_t\}$ for $t \in [a, b]$ lies entirely in $F_0 + \mathcal{K}_{\mathcal{N}}$, then $\pi(P_t)$ is constant by [Ph1] and so

$$sf(\{F_t\}) = \text{ind}(P_b P_a).$$

Thus, since the spectral flow of a path in $F_0 + \mathcal{K}_{\mathcal{N}}$ depends only on the endpoints, we will often denote it by $sf(F_a, F_b)$. All of the paths we consider in sections 3, 4, and 5 are of this kind.

2.6. Spaces Of Breuer-Fredholm Operators.

Remarks. If (\mathcal{N}, F_0) is a pre-Breuer-Fredholm module for a Banach $*$ -algebra \mathcal{A} , where $1 - F_0^2$ and $[[F_0, a]]^2$ are in the invariant operator ideal \mathcal{I} (see Appendix A), then the operators $F = uF_0u^*$ (for a dense set of unitaries in $\mathcal{U}(\mathcal{N})$) are self-adjoint Breuer-Fredholm operators in \mathcal{N} which also satisfy:

$1 - F^2 \in \mathcal{I}$, and

$F - F_0 \in \mathcal{I}^{1/2}$.

Moreover, any operator F_t in the straight line path from F_0 to $F = uF_0u^*$ also satisfies these conditions. Thus, we are led to consider the affine space of “allowable perturbations” of F_0 :

$$\mathcal{I}_{F_0} := F_0 + \{X \in \mathcal{I}_{sa}^{1/2} \mid 1 - (F_0 + X)^2 \in \mathcal{I}\} = F_0 + \{X \in \mathcal{I}_{sa}^{1/2} \mid F_0X + XF_0 \in \mathcal{I}\}.$$

We observe that if we let $P = \frac{1}{2}(\text{sign}(F_0) + 1)$, then relative to the decomposition of H determined by P ,

$$\mathcal{I}_{F_0} = \begin{pmatrix} \mathcal{I}_{sa} & \mathcal{I}_{sa}^{1/2} \\ \mathcal{I}_{sa}^{1/2} & \mathcal{I}_{sa} \end{pmatrix}.$$

In appendix B we show that for the operator ideals we are studying, this space is a real Banach space in a natural norm and $X \mapsto 1 - (F_0 + X)^2 : \mathcal{I}_{F_0} \rightarrow \mathcal{I}$ is continuous. It is in these spaces that we study the spectral flow of paths of self-adjoint Breuer-Fredholm operators.

In the case of unbounded modules, our space of “allowable perturbations” of D_0 will always be:

$$\mathcal{M}_0 = D_0 + \mathcal{N}_{sa},$$

an affine space modelled on the real Banach space \mathcal{N}_{sa} . Provided that we are careful in our choice of \mathcal{I} , (the ϵ problem) we show that the transformation $D \mapsto F = D(1 + D^2)^{-1/2}$ carries $D_0 + \mathcal{N}_{sa}$ to $F_0 + \mathcal{I}_{F_0}$, and that this transformation is suitably smooth. Thus we study the spectral flow of paths of “unbounded Breuer-Fredholm operators”, $\{D_t\}$ by considering the transformed paths $\{F_t\}$ of genuine Breuer-Fredholm operators in the bounded setting.

2.7. One-forms.

Remarks. We will consider the affine spaces defined in the previous paragraphs as real Banach manifolds, M . For example, if $M = F_0 + \mathcal{I}_{F_0}$, then for any F in M the tangent space at F is $T_F(M) = \mathcal{I}_{F_0}$.

If f is any continuous real-valued function on \mathbb{R} then provided $f : \mathcal{I} \rightarrow L^1$ is continuous, we can define a one-form, α via:

$$\alpha(X) = \frac{1}{C} \tau(Xf(1 - F^2))$$

where

$$\begin{aligned} F &\in M = F_0 + \mathcal{I}_{F_0}, \\ X &\in T_F(M) = \mathcal{I}_{F_0}, \text{ and} \\ C &= \int_{-1}^1 f(1 - t^2) dt. \end{aligned}$$

The integral of this one-form along a path $\{F_t\}$ for $t \in [0, 1]$ in M is given by

$$\frac{1}{C} \int_0^1 \tau\left(\frac{d}{dt}(F_t)f(1 - F_t^2)\right) dt.$$

To see that these integrals are independent of path, we show that such one-forms (for suitable f) are closed: that is, their exterior derivatives vanish identically. A Poincaré Lemma completes the proof of path independence.

We use the invariant definition of **exterior differentiation** [Sp]. For F in M , we have X, Y in $T_F(M) = \mathcal{I}_{F_0}$ realized as tangent vectors at F by differentiating the curves $F + sX$ and $F + sY$ at $s = 0$. That is, we also consider X and Y as the canonical vector fields on M (or flows on M) given by flowing in the X direction or Y direction. Then, by definition:

$$d\alpha(X, Y) = X \cdot (\alpha(Y)) - Y \cdot (\alpha(X)) - \alpha([X, Y]).$$

Since X and Y commute *as flows* the last term is 0 and so drops from the calculation. For the straight line paths above, this means showing that:

$$0 = d\alpha(X, Y) = \frac{1}{C} [\tau(Xf[1 - (F_0 + Y)^2]) - \tau(Yf[1 - (F_0 + X)^2])].$$

♠ Thus, to prove that our integral formulae in the bounded setting yield spectral flow (of the path) we are reduced to showing that: 1) our integrals are independent of path (by the procedure indicated above), and 2) that for certain special paths where we can actually calculate the integrals we get the desired answer. It is this second calculation that we do in the following section. ♠

3. RELATIVE INDEX OF TWO PROJECTIONS

The *essential codimension* (or *relative index*) of two projections is a fundamental tool. Formulae for this index have a long history: see for example, [ASS], [BW], [P2], [Ph1] and the references contained therein. The next result subsumes all previous ones to our knowledge.

Theorem 3.1. *Let $f : [-1, 1] \rightarrow \mathbf{R}$ be a continuous odd function with $f(1) \neq 0$. Let P and Q be projections with $P - Q \in \mathcal{K}_{\mathcal{N}}$ and $f(P - Q)$ trace class. Then $\text{ind}(QP) = \frac{1}{f(1)} \tau[f(P - Q)]$ where $\text{ind}(QP)$ is the index of QP as an operator from PH to QH .*

The proof depends on a preliminary result.

Proposition 3.2. *Let P and Q be projections on H , then the subspaces $\text{ran}P \cap \ker Q$ and $\ker P \cap \text{ran}Q$ are mutually orthogonal, closed and invariant under both P and Q . Let H_1 be the orthogonal complement of their direct sum. Then H_1 is invariant under both P and Q so that $P_1 = P|_{H_1}$ and $Q_1 = Q|_{H_1}$ are projections in $B(H_1)$ with $P_1 - Q_1 = (P - Q)|_{H_1}$, $Q_1 - P_1 = (Q - P)|_{H_1}$. Then, there exists a self-adjoint unitary U in $\{P, Q, 1\}''$ which is 1 on H_1^\perp and is such that $U(P_1 - Q_1)U^* = Q_1 - P_1$.*

Proof. We work on H_1 and observe that on this space, $\text{ran}P_1 \cap \ker Q_1$ and $\ker P_1 \cap \text{ran}Q_1$ are both $\{0\}$. Let $B = 1 - (P_1 + Q_1)$ and let $B = U|B|$ be the polar decomposition of B . Then B anticommutes with $(P_1 - Q_1)$ and so B^2 commutes with $(P_1 - Q_1)$, and hence

any continuous function of B^2 commutes with $(P_1 - Q_1)$. In particular, $|B|$ commutes with $(P_1 - Q_1)$. One easily calculates that:

$$U(P_1 - Q_1)|B| = (Q_1 - P_1)U|B|.$$

That is, $U(P_1 - Q_1)$ agrees with $(Q_1 - P_1)U$ on $\text{ran}|B|$. So, since $B = B^* = |B|U$, it suffices to see that the self-adjoint operator B has dense range (on H_1). This is equivalent to $\ker B = \{0\}$ (on H_1), which is easily seen to be equivalent to the conditions:

$$\ker P_1 \cap \text{ran} Q_1 = \{0\} = \text{ran} P_1 \cap \ker Q_1.$$

Finally, we extend U to be 1 on H_1^\perp . □

Remarks. If P, Q are as in the proposition with QP regarded as mapping PH to QH then its kernel is $\text{ran} P \cap \ker Q$ and its cokernel is the kernel of PQ on QH or, $\text{ran} Q \cap \ker P$. In particular if $P - Q$ is in \mathcal{K}_N then $QP : PH \rightarrow QH$ is a Breuer-Fredholm operator and

$$\begin{aligned} \text{ind}(QP) &= \dim(\text{ran} P \cap \ker Q) - \dim(\text{ran} Q \cap \ker P). \\ &= \tau([\text{ran} P \cap \ker Q]) - \tau([\ker P \cap \text{ran} Q]). \end{aligned}$$

We are now able to complete the proof of the theorem.

Proof. Using the notation of the previous proposition to define H_1 write

$$H = H_0 \oplus H_1 = (\text{ran} P \cap \ker Q) \oplus (\ker P \cap \text{ran} Q) \oplus H_1.$$

Then

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus P_1,$$

and

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \oplus Q_1.$$

So

$$f(P - Q) = \begin{pmatrix} f(1) & 0 \\ 0 & -f(1) \end{pmatrix} \oplus f(P_1 - Q_1).$$

Now by the previous proposition and the fact that f is odd:

$$Uf(P_1 - Q_1)U^* = -f(P_1 - Q_1).$$

By assumption this operator is trace class so we get $\tau[f(P_1 - Q_1)] = 0$. That is,

$$\tau[f(P - Q)] = f(1)\{\tau([\text{ran} P \cap \ker Q]) - \tau([\ker P \cap \text{ran} Q])\} = f(1)\text{ind}(QP)$$

by the remark. □

Remarks. At several places in this paper we consider the smooth functions, $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ of the form $f(x) = x^{-r}e^{-x^{-\sigma}}$ where $r \geq 0$ and $\sigma \geq 1$. These functions are (of course) defined to be 0 at $x = 0$.

Corollary 3.3. *If P and Q are projections and $P - Q$ is n -summable for $n \geq 1$ (not necessarily an integer) then*

$$\text{ind}(QP) = \tau[(P - Q)|P - Q|^{n-1}].$$

The case $n = 2k + 1$ an odd integer is in [ASS] and [Ph1].

Corollary 3.4. *If P and Q are projections and $P - Q$ is θ -summable (i.e., $(P - Q)^2 \in Li_0$) then,*

$$ind(QP) = e\tau[(P - Q)e^{-(P-Q)^{-2}}].$$

If $P - Q$ is weakly θ -summable (i.e., $(P - Q)^2 \in Li$) then for $\epsilon > 0$,

$$ind(QP) = e\tau[(P - Q)e^{-|P-Q|^{-(2+\epsilon)}}].$$

Proof. The first statement follows from the theorem and Corollary B.5. The second statement follows from Lemma B.4 since the function $x \mapsto e^{-tx^{-1}}$ dominates $x \mapsto e^{-x^{-(1+\delta)}}$ as $x \rightarrow \infty$ for any fixed $t > 0$ and $\delta > 0$. \square

Corollary 3.5. *If P and Q are projections and $P - Q$ is θ_q -summable (i.e., $0 < q \leq 1$ and $(P - Q)^2 \in Li_0^q$). Then, for $1 + \epsilon = 1/q$ we have*

$$ind(QP) = e\tau[(P - Q)e^{-|P-Q|^{-2(1+\epsilon)}}]$$

Proof. This follows from Corollary B.5. \square

4. THE SPECTRAL FLOW FORMULA, SIMPLEST CASE

Theorem 3.1 enables us to establish a result which serves as a prototype for all of the formulae in subsequent sections.

Theorem 4.1. *Let P and Q be infinite and co-infinite projections in the semi-finite factor \mathcal{N} and suppose $(P - Q)^2 \in Li_0^q$ for some $0 < q \leq 1$. Then $F_0 = 2P - 1$ and $F_1 = 2Q - 1$ are self-adjoint Breuer-Fredholm operators as is the path $F_t = F_0 + t(F_1 - F_0)$. Let $r \geq 0$ then*

$$sf(\{F_t\}) = \frac{1}{C_{r,q}} \int_0^1 \tau \left(\frac{d}{dt}(F_t)(1 - F_t^2)^{-r} e^{-(1-F_t^2)^{-1/q}} \right) dt$$

where

$$C_{r,q} = \int_{-1}^1 (1 - u^2)^{-r} e^{-(1-u^2)^{-1/q}} du.$$

Proof. We have

$$\begin{aligned} \frac{d}{dt}(F_t) &= F_1 - F_0 = 2(Q - P) \text{ and} \\ 1 - F_t^2 &= t(1 - t)(F_1 - F_0)^2 = 4t(1 - t)(Q - P)^2 \end{aligned}$$

and so by assumption $|1 - F_t^2|^{1/q} \in Li_0$. This means $e^{-(1-F_t^2)^{-1/q}}$ is trace class for $t \in [0, 1]$. Thus,

$$\begin{aligned} &\int_0^1 \tau \left(\frac{d}{dt}(F_t)(1 - F_t^2)^{-r} e^{-(1-F_t^2)^{-1/q}} \right) dt \\ &= \int_0^1 \tau \left[2(Q - P)[4(t - t^2)(Q - P)^2]^{-r} e^{-[4(t-t^2)(Q-P)^2]^{-1/q}} \right] dt. \end{aligned}$$

Now for each $t \in (0, 1)$ define:

$$f_t(x) = 2x[4(t - t^2)x^2]^{-r} e^{-[4(t-t^2)x^2]^{-1/q}}$$

and apply Theorem 3.1 to get

$$\begin{aligned}
& \int_0^1 \tau \left[\frac{d}{dt}(F_t)(1 - F_t^2)^{-r} e^{-(1-F_t^2)^{-1/q}} \right] dt \\
&= \int_0^1 f_t(1) \text{ind}(QP) dt \\
&= \text{ind}(QP) \int_0^1 2[4(t - t^2)]^{-r} e^{-[4(t-t^2)]^{-1/q}} dt \\
&= C_{r,q} \text{ind}(QP) \\
&= C_{r,q} sf(\{F_t\})
\end{aligned}$$

where the penultimate equality is obtained by using the change of variable $u = 2t - 1$ and the last by the definition of spectral flow [Ph1]. \square

We pause at this point to draw some conclusions from the previous analysis which we believe to be of independent interest. These results are θ -summable versions of Theorem 3.3 of [Ph1]. These results treat the case of (bounded) Breuer-Fredholm modules and do not need the full machinery of one-forms nor of the appendices. The latter are however necessary for the more general case of pre-Breuer-Fredholm modules which arise naturally when we reduce the unbounded setting to the bounded setting.

In order to emphasise the elegance of the following results we consider only the case $r = 0$ from Theorem 4.1. The case of general r is covered in the next section.

Theorem 4.2. *Let \mathcal{A} be a unital Banach $*$ -algebra and let (\mathcal{N}, F_0) be an odd θ_q -summable Breuer-Fredholm module for \mathcal{A} for some $q, 0 < q \leq 1$. Let $P = \frac{1}{2}(1 + F_0)$. For each unitary $u \in \mathcal{A}$ with $[F_0, u] \in Li_0^{q/2}$, the path $F_t^u = F_0 + t(uF_0u^* - F_0)$ lies in the self-adjoint Breuer-Fredholms and*

$$\text{ind}(PuP) = sf(\{F_t^u\}) = \frac{1}{C_{0,q}} \int_0^1 \tau \left(\frac{d}{dt}(F_t^u) e^{-|1-(F_t^u)^2|^{-1/q}} \right) dt$$

where $C_{0,q} = \int_{-1}^1 e^{-(1-u^2)^{-1/q}} du$.

Proof. The first equality follows from the discussion at the beginning of section 3 of [Ph1] and the second equality from the previous theorem with $Q = \frac{1}{2}(uF_0u^* + 1)$ since

$$P - Q = \frac{1}{2}(F_0 - uF_0u^*) = \frac{1}{2}[F_0, u]u^* \in Li_0^{q/2}.$$

\square

Corollary 4.3. *Let \mathcal{A} be a unital Banach $*$ -algebra and let (\mathcal{N}, F_0) be an odd weakly θ -summable Breuer-Fredholm module for \mathcal{A} . Let $P = \frac{1}{2}(1 + F_0)$. For each unitary $u \in \mathcal{A}$ with $[F_0, u] \in Li^{1/2}$, the path $F_t^u = F_0 + t(uF_0u^* - F_0)$ lies in the self-adjoint Breuer-Fredholms and*

$$\text{ind}(PuP) = sf(\{F_t^u\}) = \frac{1}{C_\epsilon} \int_0^1 \tau \left(\frac{d}{dt}(F_t^u) e^{-|1-(F_t^u)^2|^{-1-\epsilon}} \right) dt$$

where $C^\epsilon = \int_{-1}^1 e^{-(1-u^2)^{-1-\epsilon}} du$.

If either the module is θ -summable or if $\| [F_0, u] \|^2_{Li} < \frac{2}{3}$, then we can set $\epsilon = 0$ in the formula.

Proof. We let $q = 1/(1 + \epsilon)$, then

$$\mu_x(1 - F_0^2) \leq \frac{K}{\log x} = \frac{K}{(\log x)^{1-q}} \left(\frac{1}{\log x} \right)^q = o\left(\left(\frac{1}{\log x} \right)^q \right)$$

so that (\mathcal{N}, F_0) is θ_q -summable for \mathcal{A} . □

5. SPECTRAL FLOW FORMULAE, BOUNDED CASE

We suppose we have a θ_q -summable pre-Breuer-Fredholm module (\mathcal{N}, F_0) for the unital Banach $*$ -algebra, \mathcal{A} , where $0 < q \leq 1$.

We recall:

$$(Li_0^q)_{F_0} = \{X \in (Li_0^{q/2})_{sa} \mid F_0 X + X F_0 \in Li_0^q\}.$$

Now we set $\mathcal{M}_q = F_0 + (Li_0^q)_{F_0}$.

See appendices A and B for more details on these spaces. There we will also show that if $F_1 \in \mathcal{M}_q$ then $F_0 X + X F_0 \in Li_0^q$ if and only if $F_1 X + X F_1 \in Li_0^q$ so that the definition of \mathcal{M}_q is independent of base point. Moreover \mathcal{M}_q is contained in the self-adjoint Breuer-Fredholms so that if F_u with $u \in [0, 1]$ is a norm continuous path in this space then $sf\{F_u\}$ is well-defined as in Section 2.

Proposition 5.1. *Let (\mathcal{N}, F_0) be an odd θ_q -summable pre-Breuer-Fredholm module for the Banach $*$ -algebra \mathcal{A} . We define a one-form α_r on \mathcal{M}_q via:*

$$(\alpha_r)_F(X) = \frac{1}{C_{r,q}} \tau \left(X |1 - F^2|^{-r} e^{-|1 - F^2|^{-1/q}} \right),$$

for $F \in \mathcal{M}_q$, $X \in T_F(\mathcal{M}_q) = (Li_0^q)_{F_0}$. Then, the one-form α_r is closed (recall that $C_{r,q}$ is defined in Theorem 2 of Section 4).

Proof. By Theorem C.5 of Appendix C, the following derivative exists

$$\frac{d}{ds} \Big|_{s=0} \left\{ \tau(Y |1 - (F + sX)^2|^{-r} e^{-|1 - (F + sX)^2|^{-1/q}}) \right\}$$

and equals

$$\frac{i}{2\pi} \int_{\sigma} \tau \left\{ Y \left[g_r(T), R_{\lambda}(T) \left[[F, X]_+, T \right]_+ R_{\lambda}(T) \right]_+ \right\} m(\lambda) d\lambda$$

where $T = 1 - F^2$; where $[\cdot, \cdot]_+$ denotes the anticommutator; where $g_r(T) = |T|^{-r/2} e^{-1/2|T|^{-1/q}}$; where $R_{\lambda}(T) = (\lambda T^2 - 1)^{-1}$; and $m(\lambda) = \lambda^{r/4} e^{-(\lambda^{1/2q})/2}$.

Within the integral there are four terms, one of which is:

$$\begin{aligned}
& \tau\{Y R_\lambda(T)[F, X]_+ T R_\lambda(T) g_r(T)\} \\
&= \tau\{X T R_\lambda(T) g_r(T) Y R_\lambda(T) F\} + \tau\{X F T R_\lambda(T) g_r(T) Y R_\lambda(T)\} \\
&= \tau\{X g_r(T) R_\lambda(T) T Y F R_\lambda(T)\} + \tau\{X g_r(T) R_\lambda(T) T F Y R_\lambda(T)\} \\
&= \tau\{X g_r(T) R_\lambda(T) T[F, Y]_+ R_\lambda(T)\},
\end{aligned}$$

which is precisely one of the other terms in the integral with the roles of X and Y interchanged.

The other two terms are handled in a similar fashion. Thus

$$\begin{aligned}
& \frac{d}{ds} \Big|_{s=0} [\tau(Y | 1 - (F + sX)^2 |^{-r} e^{-|1 - (F + sX)^2|^{-1/q}})] \\
&= \frac{d}{ds} \Big|_{s=0} [\tau(X | 1 - (F + sY)^2 |^{-r} e^{-|1 - (F + sY)^2|^{-1/q}})]
\end{aligned}$$

proving the result. \square

Corollary 5.2. *If (\mathcal{N}, F_0) is a weakly θ -summable pre-Breuer-Fredholm module for \mathcal{A} and we let $\mathcal{M} = F_0 + (Li)_{F_0}$, then: for each $\epsilon > 0$ we define a one-form on \mathcal{M} by*

$$(\alpha_{r,\epsilon})_F(X) = \frac{1}{C_{r,1/(1+\epsilon)}} \tau \left(X | 1 - F^2 |^{-r} e^{-|1 - F^2|^{-1-\epsilon}} \right),$$

where $F \in \mathcal{M}$, $X \in T_F(\mathcal{M}) = (Li)_{F_0}$. Then $\alpha_{r,\epsilon}$ is closed. Here

$$C_{r,1/(1+\epsilon)} = \int_{-1}^1 (1 - u^2)^{-r} e^{-(1-u^2)^{-1-\epsilon}} du.$$

If the module is θ -summable then the conclusion holds on $\mathcal{M}_1 = F_0 + (Li_0)_{F_0}$ for the one-form $\alpha_{r,0}$.

Proof. With $q = 1/(1 + \epsilon)$ this follows from the fact that the inclusion of Li into Li_0^q is bounded by Lemma A.5. \square

Definition 5.3. Fix (\mathcal{N}, F_0) with $1 - F_0^2 \in Li_0^q$. Let $F \in \mathcal{M}_q$ and let $F_t = F_0 + t(F - F_0)$ for $t \in [0, 1]$ be the linear path in \mathcal{M}_q from F_0 to F . Define $\theta_r : \mathcal{M}_q \rightarrow \mathbb{R}$ for $r \geq 0$ via:

$$\theta_r(F) = \frac{1}{C_{r,q}} \int_0^1 \tau \left[(F - F_0) | 1 - F_t^2 |^{-r} e^{-|1 - F_t^2|^{-1/q}} \right] dt.$$

Recall that by definition

$$d\theta_F(X) = \frac{d}{ds} \Big|_{s=0} (\theta(F + sX))$$

for $X \in T_F(\mathcal{M}_q)$.

Proposition 5.4. (Poincaré Lemma) With the above definitions, $d\theta_r = \alpha_r$.

Proof. Fix $F \in \mathcal{M}_q$ and let $Y = F - F_0$. For $T \in Li_0^q$, let $f_r(T) = |T|^{-r} e^{-|T|^{-1/q}}$ so that

$$\theta_r(F) = \frac{1}{C_{r,q}} \int_0^1 \tau [Y f_r(1 - F_t^2)] dt$$

where $F_t = F_0 + tY$ for $t \in [0, 1]$. Let $X \in (Li_0^q)_{F_0}$ so that

$$\theta_r(F + sX) = \frac{1}{C_{r,q}} \int_0^1 \tau [(Y + sX) f_r(1 - (F_t^s)^2)] dt$$

where $F_t^s = F_0 + t(F + sX - F_0) = F_0 + t(Y + sX) = F_t + s(tX)$ for real s . Now by the product rule

$$\frac{d}{ds} \tau[(Y + sX) f_r(1 - (F_t^s)^2)] = \tau[(X) f_r(1 - (F_t^s)^2)] + \tau[(Y + sX) \frac{d}{ds} f_r(1 - (F_t^s)^2)].$$

By Theorem C.5 of Appendix C with F_t in place of F_0 , tX in place of X , and an additive change of the variable s we get:

$$\begin{aligned} & \frac{d}{ds} (f_r(1 - (F_t^s)^2)) \\ &= \frac{i}{2\pi} \left\{ \int_{\sigma} [f_r(T_{t,s}), (\lambda T_{t,s}^2 - 1)^{-1} [(F_t^s tX + tX F_t^s), T_{t,s}]_+ (\lambda T_{t,s}^2 - 1)^{-1}]_+ \lambda^{r/4} e^{-\lambda^{1/2q}/2} d\lambda \right\} \end{aligned}$$

where $T_{t,s} = 1 - (F_t^s)^2$ and $[\cdot, \cdot]_+$ denotes the anti-commutator. Similar estimates to those of the proof of Theorem C.5 show that

$$(t, s) \mapsto \frac{d}{ds} \tau [(Y + sX) f_r(1 - (F_t^s)^2)]$$

is continuous on $[0, 1] \times \mathbb{R}$. Since

$$(t, s) \mapsto \tau [(Y + sX) f_r(1 - (F_t^s)^2)]$$

is also continuous, Theorem 11 of Chapter X of [L], allows us to differentiate under the integral sign to compute

$$\begin{aligned} & \frac{d}{ds} \Big|_{s=0} \int_0^1 \tau [(Y + sX) f_r(1 - (F_t^s)^2)] dt \\ &= \int_0^1 \left\{ \tau [X f_r(1 - F_t^2)] + \tau \left[Y \frac{d}{ds} \Big|_{s=0} f_r(1 - (F_t^s)^2) \right] \right\} dt \end{aligned} \quad (**).$$

Now

$$\begin{aligned} \tau \left[Y \frac{d}{ds} \Big|_{s=0} f_r(1 - (F_t^s)^2) \right] &= \tau \left[tX \frac{d}{ds} \Big|_{s=0} f_r(1 - (F_t + sY)^2) \right] \\ &= t\tau \left[X \frac{d}{ds} \Big|_{s=0} f_r(1 - (F_{t+s})^2) \right] = t\tau \left[X \frac{d}{dt} f_r(1 - F_t^2) \right]. \end{aligned}$$

Substitution in (**) and integration by parts gives, for the RHS of (**)

$$\int_0^1 \frac{d}{dt} [\tau (X f_r(1 - F_t^2))] dt = \tau (X f_r(1 - F^2))$$

as $F_1 = F$. Dividing by the normalisation constant gives the result. \square

Corollary 5.5. *The integral of the one-form α_r along a piecewise C^1 path Γ in \mathcal{M}_q depends only on the endpoints of the path Γ .*

Proof. This follows as in Proposition 1.5 of [CP1]. \square

Definition 5.6. Fix F_0 a self-adjoint Breuer-Fredholm with $1 - F_0^2 \in Li_0^q$. Let $F \in \mathcal{M}_q$ and recall from Section 2 that $\tilde{F} = \text{sign}(F) \in \mathcal{M}_q$. Let $\{F_t\}$ for $t \in [0, 1]$ be a C^1 path in \mathcal{M}_q beginning at F and ending at \tilde{F} . For example $F_t = F + t(\tilde{F} - F)$ will do. Define

$$\gamma_{r,q}(F) = \frac{1}{C_{r,q}} \int_0^1 \tau \left[\frac{d}{dt}(F_t) f_r(1 - F_t^2) \right] dt.$$

It follows by the previous corollary that $\gamma_{r,q}$ is well-defined. Moreover, it is clear by considering the linear path that if F_1 and F_2 are unitarily equivalent in \mathcal{M}_q then $\gamma_{r,q}(F_1) = \gamma_{r,q}(F_2)$.

Theorem 5.7. Let \mathcal{A} be a unital Banach $*$ -algebra and let (\mathcal{N}, F_0) be an odd θ_q -summable pre-Breuer-Fredholm module for \mathcal{A} for some q , $0 < q \leq 1$. Let $r \geq 0$ then if $F_j \in \mathcal{M}_q$ for $j = 1, 2$, the spectral flow along any piecewise C^1 path $\{F_t\}$ in \mathcal{M}_q , $t \in [1, 2]$ joining F_1 and F_2 is given by

$$sf(F_1, F_2) = \frac{1}{C_{r,q}} \int_1^2 \tau \left[\frac{d}{dt}(F_t) f_r(1 - F_t^2) \right] dt + \gamma_{r,q}(F_2) - \gamma_{r,q}(F_1)$$

where $f_r(T) = |T|^{-r} e^{-|T|^{-1/q}}$.

Proof. The formula on the right is just the integral of α_r along the piecewise C^1 path in \mathcal{M}_q from \tilde{F}_1 to \tilde{F}_2 made up of the three parts, \tilde{F}_1 to F_1 , then $\{F_t\}$ for $t \in [1, 2]$ and finally F_2 to \tilde{F}_2 . But since the integral of α_r is independent of the path in \mathcal{M}_q , we may equally use the straight line path joining \tilde{F}_1 and \tilde{F}_2 . Then we have by Theorem 4.1:

$$sf(\tilde{F}_1, \tilde{F}_2) = \frac{1}{C_{r,q}} \int_1^2 \tau \left[\frac{d}{dt}(F_t) f_r(1 - F_t^2)^{-1} \right] dt + \gamma_{r,q}(F_2) - \gamma_{r,q}(F_1).$$

Finally,

$$\begin{aligned} sf(\tilde{F}_1, \tilde{F}_2) &= sf(\tilde{F}_1, F_1) + sf(F_1, F_2) + sf(F_2, \tilde{F}_2) \\ &= sf(F_1, F_2) \end{aligned}$$

as there is no spectral flow along the paths joining F_j and \tilde{F}_j as noted in the proof of Theorem 1.7, p.683 of [CP1]. \square

Corollary 5.8. If we assume that (\mathcal{N}, F_0) is an odd weakly θ -summable pre-Breuer-Fredholm module for \mathcal{A} . Then for

$$F_j \in \mathcal{M} = F_0 + (Li)_{F_0}, \quad j = 1, 2,$$

and the remaining hypotheses intact we get:

$$sf(F_1, F_2) = \frac{1}{C^\epsilon} \int_1^2 \tau \left[\frac{d}{dt}(F_t) e^{-|1 - F_t^2|^{-1-\epsilon}} \right] dt + \gamma^\epsilon(F_2) - \gamma^\epsilon(F_1)$$

(recall $C^\epsilon = \int_{-1}^1 e^{-(1-u^2)^{-1-\epsilon}} du$) and

$$\gamma^\epsilon(F) = \frac{1}{C^\epsilon} \int_0^1 \tau \left[(\tilde{F} - F) e^{-|1 - F_t^2|^{-1-\epsilon}} \right] dt$$

where this last integral is along the linear path from F to $\tilde{F} = \text{sign}(F)$.

Again, if the module is θ -summable we can take $\epsilon = 0$ on $\mathcal{M}_1 = F_0 + (Li)_{F_0}$.

Proof. This follows from the fact that the inclusion of Li in Li_0^q for $q = 1/(1+\epsilon)$ is bounded by Lemma A.5. \square

Theorem 5.9. *Let (\mathcal{N}, F_0) be an odd θ_q -summable pre-Breuer-Fredholm module for the unital Banach $*$ -algebra \mathcal{A} and for some q , $0 < q \leq 1$. Let $P = \frac{1}{2}(\text{sign}(F_0) + 1)$. For each unitary u in \mathcal{A} with $[F_0, u] \in Li_0^{q/2}$ the path $F_t^u = F_0 + t(uF_0u^* - F_0)$ lies in $\mathcal{M}_q = F_0 + (Li_0^q)_{F_0}$ and*

$$\text{ind}(PuP) = sf(\{F_t^u\}) = \frac{1}{C_{r,q}} \int_0^1 \tau \left[\frac{d}{dt}(F_t^u) f_r(1 - (F_t^u)^2) \right] dt,$$

where $f_r(T) = |T|^{-r} e^{-|T|^{-1/q}}$ for $r \geq 0$.

Proof. The second equality follows from the previous theorem since

$$\gamma_{r,q}(uF_0u^*) = \gamma_{r,q}(F_0).$$

The first equality follows from the discussion at the beginning of section 3 of [Ph1]. \square

An analogue of Corollary 5.8 follows from Theorem 5.9 in the obvious fashion.

6. PRELIMINARIES FOR THE UNBOUNDED CASE

Lemma 6.1 (cf [CP1] Lemma 6, Appendix B). *If D_0 is an unbounded self-adjoint operator, A a bounded self-adjoint operator and $D = D_0 + A$ then*

- (1) $(1 + D^2)^{-1} \leq f(\|A\|)(1 + D_0^2)^{-1}$ and
- (2) $-(f(\|A\|) - 1)(1 + D_0^2)^{-1} \leq (1 + D^2)^{-1} - (1 + D_0^2)^{-1} \leq (f(\|A\|) - 1)(1 + D_0^2)^{-1}$ where $f(a) = 1 + \frac{1}{2}(a^2 + a\sqrt{a^2 + 4})$.

Proof. The first result is the one cited. For (2) the right hand inequality follows from (1) by subtracting $(1 + D_0^2)^{-1}$. The left hand inequality of (2) follows by noting first that $(1 + D_0^2)^{-1} \leq f(\|A\|)(1 + D^2)^{-1}$ by interchanging the roles of D and D_0 . Then

$$\left(\frac{1}{f(\|A\|)} - 1 \right) (1 + D_0^2)^{-1} \leq (1 + D^2)^{-1} - (1 + D_0^2)^{-1}$$

or

$$-\frac{(f(\|A\|) - 1)}{f(\|A\|)} (1 + D_0^2)^{-1} \leq (1 + D^2)^{-1} - (1 + D_0^2)^{-1}.$$

Since $-(f(\|A\|) - 1) \leq -\frac{(f(\|A\|) - 1)}{f(\|A\|)}$ we are done. \square

Lemma 6.2 ([CP1] Lemma 2.7). *If $D = D_0 + A$ with $A \in \mathcal{N}_{sa}$ and with D_0 affiliated to \mathcal{N} we define $F_A = D(1 + D^2)^{-1/2}$ and $F_0 = D_0(1 + D_0^2)^{-1/2}$ then for $0 < \sigma < 1$, $(F_A - F_0)^2 = B_\sigma(1 + D_0^2)^{\sigma/2}$ where $B_\sigma \in \mathcal{N}$ and $\|B_\sigma\| \leq C(\sigma)\|A\|$ where $C(\sigma)$ is a constant depending only on D_0 and σ .*

Corollary 6.3. *Let (\mathcal{N}, D_0) be weakly θ -summable and let $D_t = D_0 + A_t \in D_0 + \mathcal{N}_{sa}$ be an operator-norm continuous path. Then for fixed q , $0 < q < 1$*

$$t \mapsto F_t = D_t(1 + D_t^2)^{-1/2} \in F_{D_0} + (Li_0^q)_{F_{D_0}}$$

is continuous, where $F_{D_0} = D_0(1 + D_0^2)^{-1/2}$.

Proof. Let $q < \sigma < 1$. Since $(1 + D_0^2)^{-1} \in Li$ by definition we have $X_t = F_t - F_{D_0} \in Li^{\sigma/2} \subseteq Li_0^{q/2}$ by Lemma 6.2. Using Lemma 6.2 again with q in place of σ we get

$$\|X_t - X_{t_0}\|_{Li_0^{q/2}} \leq C(q)\|A_t - A_{t_0}\| \cdot \|(1 + D_{t_0}^2)^{-q/2}\|_{Li_0^{q/2}} \rightarrow 0$$

as $t \rightarrow t_0$ (noting that $(1 + D_{t_0}^2)^{-q/2} \in Li_0^{q/2}$ by part (1) of Lemma 6.1). Thus $F_t \in F_{D_0} + Li_0^{q/2}$ and is continuous in that space. Now $t \rightarrow 1 - (F_{D_0} + X_t)^2 = 1 - F_t^2 = (1 + D_t^2)^{-1}$ is in $Li \subseteq Li_0^{q/2}$ by part (1) of Lemma 6.1. Moreover it is continuous in Li_0^q by part (2) and the fact that Li_0^q is an invariant operator ideal (see Appendix A). The result now follows from part (3), Lemma B.12. \square

Proposition 6.4. *Let D_0 be an unbounded self-adjoint operator affiliated with \mathcal{N} . If \mathcal{I} is an invariant operator ideal in \mathcal{N} , $(1 + D_0^2)^{-1} \in \mathcal{I}$ and $t \mapsto D_t = D_0 + A_t \in D_0 + \mathcal{N}_{sa}$ is a C^1 path in the operator norm and $F_t = D_t(1 + D_t^2)^{-1/2}$ then $t \mapsto 1 - F_t^2$ is C^1 in \mathcal{I} .*

Proof. By part (1) of Lemma 6.1 we have $1 - F_t^2 = (1 + D_t^2)^{-1} \in \mathcal{I}$ and depends continuously on t by part (2). Using Lemma 2.9 of [CP1] (in the notation used there we set $x = 1$) we have

$$\begin{aligned} & \frac{1}{t - t_0} [(1 + D_t^2)^{-1} - (1 + D_{t_0}^2)^{-1}] \\ &= -D_{t_0}(1 + D_{t_0}^2)^{-1} \frac{A_t - A_{t_0}}{t - t_0} (1 + D_t^2)^{-1} - (1 + D_{t_0}^2)^{-1} \frac{A_t - A_{t_0}}{t - t_0} D_t(1 + D_t^2)^{-1}. \end{aligned}$$

Now, by assumption $\frac{A_t - A_{t_0}}{t - t_0} \rightarrow A'_{t_0}$ in operator norm and $(1 + D_t^2)^{-1} \rightarrow (1 + D_{t_0}^2)^{-1}$ in \mathcal{I} -norm by continuity and so the first term converges to

$$D_{t_0}(1 + D_{t_0}^2)^{-1} A'_{t_0} (1 + D_{t_0}^2)^{-1}$$

in \mathcal{I} -norm. For the second term we note that $(1 + D_{t_0}^2)^{-1}$ is fixed in \mathcal{I} and the rest converges in operator norm to $A'_{t_0} D_{t_0}(1 + D_{t_0}^2)^{-1}$ by Corollary 2 of Appendix A of [CP1]. Thus the derivative of $1 - F_t^2$ exists in \mathcal{I} -norm and equals

$$D_t(1 + D_t^2)^{-1} A'_t (1 + D_t^2)^{-1} - (1 + D_t^2)^{-1} A'_t D_t(1 + D_t^2)^{-1}.$$

As a function of t this is continuous in \mathcal{I} -norm as $t \mapsto (1 + D_t^2)^{-1}$ is \mathcal{I} -norm continuous and the rest is operator norm continuous. \square

Proposition 6.5 (cf Proposition 2.10 of [CP1]). *Let (\mathcal{N}, D_0) be weakly θ -summable and let $t \mapsto A_t$ be a C^1 path in \mathcal{N}_{sa} then with $D_t = D_0 + A_t$ we have that $t \mapsto F_t = D_t(1 + D_t^2)^{-1/2}$ is a path of Breuer-Fredholm operators in $F_{D_0} + (Li_0^q)_{F_{D_0}}$ for $0 < q < 1$ which is C^1 in the norm on that space. Moreover*

$$\begin{aligned} \frac{d}{dt} F_t &= \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} [(1 + D_t^2 + \lambda)^{-1} (1 + \lambda) A'_t (1 + D_t^2 + \lambda)^{-1} \\ &\quad - D_t(1 + D_t^2 + \lambda)^{-1} A'_t D_t(1 + D_t^2 + \lambda)^{-1}] d\lambda \end{aligned}$$

where the integral converges in the $Li_0^{q/2}$ -norm and hence also in operator norm.

Proof. The convergence of the integral follows because each $(1 + D_t^2)^{-1}$ is in Li by Lemma 6.1 and hence in Li_0^q for any $q < 1$. Thus $(1 + D_t^2)^{-q/2}$ is in $Li_0^{q/2}$. Now we use the proof of Proposition 2.10 of [CP1] by setting, in the notation used there, $\frac{q}{2} = \frac{1}{2} - \epsilon$ and replacing the \mathcal{L}^q -norm by the $Li_0^{q/2}$ -norm. Then the proofs of convergence and continuity go through verbatim as they depend only on operator norm estimates and the fact that the ideals in question are invariant operator ideals.

The same remarks apply to the proof of the existence of $\frac{d}{dt}F_t$ in the $Li_0^{q/2}$ -norm and the integral formula. To see that the derivative is continuous in $Li_0^{q/2}$ -norm we need only note that $(1 + D_t^2)^{-q/2} \leq f(\|A_t\|)^{q/2}(1 + D_0^2)^{-q/2}$ by Lemma 6.1 and operator monotonicity so that the result follows by our previous remarks (and $\|(1 + D_t^2)^{-q/2}\|_{Li_0^{q/2}} \leq C\|(1 + D_0^2)^{-q/2}\|_{Li_0^{q/2}}$ for all t).

Finally, for the proof that $t \mapsto F_t$ is C^1 in the norm of $F_{D_0} + (Li_0^q)_{F_{D_0}}$ we first use Corollary 6.1 to see that it is at least continuous there. Then we apply Proposition 6.1 with $\mathcal{I} = Li_0^q$ to see that $t \mapsto 1 - F_t^2$ is C^1 in Li_0^q (notice that this does not follow from the product rule as the F_t themselves are not in $Li_0^{q/2}$). The result now follows by Lemma B.15. \square

Remarks. Using Lemma B.15 in exactly the same fashion, we can immediately improve the conclusion of Proposition 2.10 of [CP1] to read (in the notation of [CP1]) that $t \mapsto F_t = D_t(1 + D_t^2)^{-1/2}$ is C^1 in $F_0 + \mathcal{L}_{sa}^{q, \frac{q}{2}}$. In the notation of this paper, the latter space is denoted $F_0 + (\mathcal{L}^q)_{F_0}$.

Remark 1.8 of [CP1] is now unnecessary.

Proposition 6.6. *Let (\mathcal{N}, D_0) be weakly θ -summable and let $t \mapsto A_t$ be a C^1 path in \mathcal{N}_{sa} then with $D_t = D_0 + A_t$ and $F_t = D_t(1 + D_t^2)^{-1/2}$ we have for $0 < q < 1$ that the maps*

$$t \mapsto F_t'|1 - F_t^2|^{-3/2}e^{-|1 - F_t^2|^{-1/q}}$$

and

$$t \mapsto D_t'e^{-(1 + D_t^2)^{1/q}}$$

are both continuous \mathcal{L}^1 -valued functions of t and

$$\tau \left(F_t'|1 - F_t^2|^{-3/2}e^{-|1 - F_t^2|^{-1/q}} \right) = \tau \left(D_t'e^{-(1 + D_t^2)^{1/q}} \right)$$

for all t .

Proof. By Corollary 6.3, $t \mapsto 1 - F_t^2 \in Li_0^q$ is continuous. Then, by Corollary B.10

$$t \mapsto |1 - F_t^2|^{-3/2}e^{-|1 - F_t^2|^{-1/q}} \in \mathcal{L}^1$$

is continuous. Apply Proposition 6.5 to see that

$$t \mapsto F_t'|1 - F_t^2|^{-3/2}e^{-|1 - F_t^2|^{-1/q}} \in \mathcal{L}^1$$

is continuous. Now since

$$t \mapsto (1 + D_t^2)^{-1} = 1 - F_t^2 \in Li_0^q$$

is continuous we have $t \mapsto e^{-(1+D_t^2)^{1/q}} \in \mathcal{L}^1$ is continuous by Corollary B.9. Thus $t \mapsto D_t' e^{-(1+D_t^2)^{1/q}} \in \mathcal{L}^1$ is continuous. Finally the last claim of the Proposition follows by using the integral formula for F_t' of Proposition 6.5 and then multiplying through by $e^{-(1+D_t^2)^{1/q}} = e^{-|1-F_t^2|^{-1/q}}$ to get an integral converging in trace-norm. One then passes the trace through the integral and uses the cyclicity of the trace so as to allow an application of Lemma 2.11 of [CP1]. \square

7. SPECTRAL FLOW FORMULAE, UNBOUNDED CASE

We begin by noting the following result.

Proposition 7.1. *Assume that (\mathcal{N}, D_0) is an odd unbounded weakly θ -summable Breuer-Fredholm module for the Banach $*$ -algebra \mathcal{A} and let $F_0 = D_0(1 + D_0^2)^{-1/2}$. Then (\mathcal{N}, F_0) is an odd θ_q -summable pre-Breuer-Fredholm module for \mathcal{A} if $0 < q < 1$.*

Proof. This follows by an argument similar to that in Proposition 2.4 of [CP1] or alternatively as in [CPS]. \square

Definition 7.2. *Let (\mathcal{N}, D_0) be an odd unbounded weakly θ -summable (respectively, θ -summable) Breuer-Fredholm module for the Banach $*$ -algebra \mathcal{A} . Let $\mathcal{M}_0 = D_0 + \mathcal{N}_{sa}$ and for $D \in \mathcal{M}_0$ and $X \in T_D(\mathcal{M}_0) = \mathcal{N}_{sa}$, the tangent space to \mathcal{M}_0 at D , the map*

$$\alpha_q(X) = \frac{1}{C_q} \tau \left(X e^{-(1+D^2)^{1/q}} \right)$$

defines the one-form α_q on \mathcal{M}_0 for $0 < q < 1$ (respectively, $0 < q \leq 1$) where $C_q = C_{0,q} = \int_{-\infty}^{\infty} e^{-(1+x^2)^{1/q}} dx$. (We note that $C_1 = C_{0,1} = \int_{-\infty}^{\infty} e^{-(1+x^2)} dx = \frac{\sqrt{\pi}}{e}$)

We recall from the definitions of Section 5 that if $F \in F_{D_0} + (Li_0^q)_{F_{D_0}}$ then

$$\gamma_{\frac{3}{2},q}(F) = \frac{1}{C_{\frac{3}{2},q}} \int_0^1 \tau \left(F_t' |1 - F_t^2|^{-3/2} e^{-|1-F_t^2|^{-1/q}} \right) dt$$

where $\{F_t\}$ is the linear path from F to $\text{sign} F$. We note that

$$C_{\frac{3}{2},q} = \int_{-1}^1 (1-u^2)^{-3/2} e^{-(1-u^2)^{-1/q}} du = \int_{-\infty}^{\infty} e^{-(1+x^2)^{1/q}} dx = C_q.$$

Lemma 7.3. *Let $\beta(X) = \tau(Xg(D))$, $X \in \mathcal{N}_{sa} = T_D(\mathcal{M}_0)$ for $D \in \mathcal{M}_0$ be a one-form where $g : \mathcal{M}_0 \rightarrow \mathcal{L}^1$ is continuous and the integral of β is independent of the piecewise C^1 path in \mathcal{M}_0 . Then $\beta = df$ where $f(D) = \int_0^1 \tau(D_t' g(D_t)) dt$ and $\{D_t\}$ is any such path in \mathcal{M}_0 from D_0 to D . That is, β is exact.*

Proof. Recall $df_D(X) = \frac{d}{ds}|_{s=0}(f(D+sX))$. For each s choose our path from D_0 to $D+sX$ to pass through D and be linear from D to $D+sX$ and be indexed by $r \in [0, s]$ (or $[s, 0]$ if $s < 0$). Then $f(D+sX) = f(D) + \int_0^s \tau(Xg(D+rX)) dr$ and therefore

$$\frac{d}{ds}|_{s=0}(f(D+sX)) = \tau(Xg(D)) = \beta(X)$$

as claimed. \square

Theorem 7.4. *Let (\mathcal{N}, D_0) be an odd unbounded weakly θ -summable Breuer-Fredholm module for the Banach $*$ -algebra \mathcal{A} . Let $\mathcal{M}_0 = D_0 + \mathcal{N}_{sa}$ then for $0 < q < 1$ the integral of the one-form*

$$\alpha_q(X) = \frac{1}{C_q} \tau \left(X e^{-(1+D^2)^{1/q}} \right)$$

is independent of the path in \mathcal{M}_0 and hence α_q is exact. Moreover, if $\{D_t\}_{t \in [a,b]}$ is any piecewise C^1 path in \mathcal{M}_0 then

$$sf(D_a, D_b) = \frac{1}{C_q} \int_a^b \tau \left(D'_t e^{-(1+D_t^2)^{1/q}} \right) dt + \gamma_{\frac{3}{2},q}(D_b(1+D_b^2)^{-1/2}) - \gamma_{\frac{3}{2},q}(D_a(1+D_a^2)^{-1/2}).$$

Proof. We know that with $F_0 = D_0(1+D_0^2)^{-1/2}$, (\mathcal{N}, F_0) is θ_q -summable by Proposition 7.1. Moreover $t \mapsto F_t = D_t(1+D_t^2)^{-1/2}$ is a piecewise C^1 path in $F_0 + (Li_0^q)_{F_0}$ by Proposition 6.5. Now we recall from definition 2.15 of [CP1] that $sf(D_a, D_b) = sf(F_a, F_b)$ and so by Theorem 5.1 (with $r = 3/2$) together with Proposition 6.6 we obtain our formula. It follows from this formula that the integral of the one-form α_q is independent of the path in \mathcal{M}_0 . \square

Corollary 7.5. *Let (\mathcal{N}, D_0) be an odd unbounded weakly θ -summable Breuer-Fredholm module for the Banach $*$ -algebra \mathcal{A} , and let $0 < q < 1$. Let $P = \chi_{[0,\infty)}(D_0)$, then for each unitary $u \in \mathcal{A}$ with $u(\text{dom } D_0) \subseteq \text{dom}(D_0)$ and $[D_0, u]$ bounded we have that PuP is a Breuer-Fredholm operator in PNP and if $\{D_t^u\}$ is any piecewise C^1 path in $\mathcal{M}_0 = D_0 + \mathcal{N}_{sa}$ from D_0 to uD_0u^* (for example the linear path) then*

$$\text{ind}(PuP) = sf(\{D_t^u\}) = \frac{1}{C_q} \int_0^1 \tau \left((D_t^u)' e^{-(1+(D_t^u)^2)^{1/q}} \right) dt.$$

Proof. Since

$$uD_0u^* = D_0 - [D_0, u]u^* \in D_0 + \mathcal{N}_{sa}$$

the right hand equality follows from the previous theorem since

$$\gamma_{\frac{3}{2},q}(uD_0(1+D_0^2)^{-1/2}u^*) = \gamma_{\frac{3}{2},q}(D_0(1+D_0^2)^{-1/2}).$$

For $F_0 = D_0(1+D_0^2)^{-1/2}$ we have that (\mathcal{N}, F_0) is θ_q -summable by Proposition 7.1 and so by Theorem 5.9 $\text{ind}(PuP)$ is given by the spectral flow along the linear path from F_0 to uF_0u^* and hence is the spectral flow of any piecewise C^1 path from F_0 to uF_0u^* . In particular,

$$\text{ind}(PuP) = sf(\{D_t^u(1+(D_t^u)^2)^{-1/2}\}) = sf(\{D_t^u\})$$

\square

Remarks. Let (\mathcal{N}, D_0) be θ -summable (for \mathbb{C} say) and for each $D \in \mathcal{M}_0 = D_0 + \mathcal{N}_{sa}$ we let $F_D = D(1+D^2)^{-1/2}$. Then $\gamma_{\frac{3}{2},1}(F_D)$ is well-defined even though we do not know whether $F_D \in F_{D_0} + (Li_0)_{F_{D_0}}$. This follows from the fact that $1 - F_D^2 = (1+D^2)^{-1} \in Li_0$ by Lemma 6.1, and that the definition of $\gamma_{\frac{3}{2},1}$ in Section 5 only involves F_D and $\tilde{F} = \text{sign}(F_D)$.

♠ The following lemma finally allows us to get rid of the annoying q in our formula when our module is actually θ -summable. ♠

Lemma 7.6. *Let (\mathcal{N}, D_0) be an odd unbounded θ -summable Breuer-Fredholm module for the Banach $*$ -algebra \mathcal{A} , and let $\{D_t\}_{t \in [a, b]}$ be a piecewise C^1 path in \mathcal{M}_0 . Then, for each $D \in \mathcal{M}_0$ where $F_D = D(1 + D^2)^{-1/2}$,*

$$\lim_{q \rightarrow 1^-} \int_0^1 \tau \left(D'_t e^{-(1+D_t^2)^{1/q}} \right) dt = \int_0^1 \tau \left(D'_t e^{-(1+D_t^2)} \right) dt, \quad (7.1)$$

$$\lim_{q \rightarrow 1^-} \gamma_{\frac{3}{2}, q}(F_D) = \gamma_{\frac{3}{2}, 1}(F_D), \quad (7.2)$$

$$\lim_{q \rightarrow 1^-} C_q = C_1. \quad (7.3)$$

Proof. The proof rests on a subsidiary result. Suppose that $X \in Li_0$ and $|X| \leq 1$. Then, for $0 < q < 1$, $0 \leq e^{-|X|^{-1/q}} \leq e^{-|X|^{-1}}$ and so

$$0 \leq |X|^{-r} e^{-|X|^{-1/q}} \leq |X|^{-r} e^{-|X|^{-1}}$$

for any $r \geq 0$. By Corollary B.11 these operators are in \mathcal{L}^1 and so

$$\begin{aligned} & \| |X|^{-r} e^{-|X|^{-1}} - |X|^{-r} e^{-|X|^{-1/q}} \|_1 \\ &= \tau(|X|^{-r} e^{-|X|^{-1}} - |X|^{-r} e^{-|X|^{-1/q}}) \\ &= \tau(|X|^{-r} e^{-|X|^{-1}}) - \tau(|X|^{-r} e^{-|X|^{-1/q}}) \\ &= \int_0^\infty \mu_s(X)^{-r} \left[e^{-\mu_s(X)^{-1}} - e^{-\mu_s(X)^{-1/q}} \right] ds \end{aligned}$$

by [FK] Remark 3.3. Now the integrand converges pointwise to 0 as $q \rightarrow 1^-$ and is dominated by the integrable function $s \mapsto \mu_s(X)^{-r} e^{-\mu_s(X)^{-1}}$. Thus the integral converges to 0 as $q \rightarrow 1^-$. That is

$$\lim_{q \rightarrow 1^-} \| |X|^{-r} e^{-|X|^{-1}} - |X|^{-r} e^{-|X|^{-1/q}} \|_1 = 0 \quad (7.4).$$

Now, to see (7.1):

$$\begin{aligned} & \left| \int_0^1 \tau \left(D'_t e^{-(1+D_t^2)} \right) dt - \int_0^1 \tau \left(D'_t e^{-(1+D_t^2)^{1/q}} \right) dt \right| \\ & \leq \int_0^1 \| D'_t \|_1 \| e^{-(1+D_t^2)} - e^{-(1+D_t^2)^{1/q}} \|_1 dt \leq C \int_0^1 \| e^{-(1+D_t^2)} - e^{-(1+D_t^2)^{1/q}} \|_1 dt. \end{aligned}$$

By (7.4) with $X = (1 + D_t^2)^{-1}$ and $r = 0$ we see that the integrand goes to 0 pointwise in t . However

$$\| e^{-(1+D_t^2)} - e^{-(1+D_t^2)^{1/q}} \|_1 = \tau \left(e^{-(1+D_t^2)} - e^{-(1+D_t^2)^{1/q}} \right) \leq \tau \left(e^{-(1+D_t^2)} \right).$$

But

$$t \mapsto (1 + D_t^2)^{-1} \mapsto e^{-(1+D_t^2)} \in \mathcal{L}^1$$

is continuous by Lemma 6.1 and Corollary B.11. Hence, $t \mapsto \tau \left(e^{-(1+D_t^2)} \right)$ is integrable.

Thus,

$$\int_0^1 \| e^{-(1+D_t^2)} - e^{-(1+D_t^2)^{1/q}} \|_1 dt \rightarrow 0$$

as $q \rightarrow 1^-$ and (7.1) follows.

Next we observe that (7.3) is an easy application of the dominated convergence theorem as $C_q = \int_{-\infty}^{\infty} e^{-(1+x^2)^{1/q}} dx$. Since $C_q = C_{\frac{3}{2},q}$ we also get $\lim_{q \rightarrow 1^-} C_{\frac{3}{2},q} = C_{\frac{3}{2},1}$.

Finally, we obtain (7.2) by a method very similar to that used for (7.1) using (7.3) and recalling that

$$\gamma_{\frac{3}{2},q}(F_D) = \frac{1}{C_{\frac{3}{2},q}} \int_0^1 \tau \left(F'_t |1 - F_t^2|^{-3/2} e^{-|1 - F_t^2|^{-1/q}} \right) dt$$

where $F_t = (1-t)F_D + t\tilde{F}_D$ is the linear path and $1 - F_t^2 \in Li_0$ with $|1 - F_t^2| \leq 1$. \square

Definition 7.7. Let (\mathcal{N}, D_0) be an odd unbounded θ -summable Breuer-Fredholm module for the Banach $*$ -algebra \mathcal{A} and let $D \in \mathcal{M}_0$. Also let

$$\gamma_0(D) = \gamma_{\frac{3}{2},1}(D(1 + D^2)^{-1/2})$$

and for $X \in T_D(\mathcal{M}_0) = \mathcal{N}_{sa}$ define a one-form α on \mathcal{M}_0 by

$$\alpha(X) = \frac{1}{C_1} \tau \left(X e^{-(1+D^2)} \right) = \frac{1}{\sqrt{\pi}} \tau \left(X e^{-D^2} \right).$$

Theorem 7.8. Let (\mathcal{N}, D_0) be an odd unbounded θ -summable Breuer-Fredholm module for the Banach $*$ -algebra \mathcal{A} . Then the integral of the one-form α is independent of the path in \mathcal{M}_0 so that α is exact and moreover if $\{D_t\}_{t \in [a,b]}$ is any piecewise C^1 path in \mathcal{M}_0 then

$$sf(D_a, D_b) = \frac{1}{\sqrt{\pi}} \int_a^b \tau \left(D'_t e^{-D_t^2} \right) dt + \gamma_0(D_b) - \gamma_0(D_a).$$

Proof. As θ -summable implies weakly θ -summable the last formula follows from Theorem 7.4, Lemma 7.6 and the definition preceding the theorem. That the integral of α is independent of the path in \mathcal{M}_0 now follows from this formula. \square

Remark: Note that the methods of Appendix C may also be used to prove directly that α is closed. Then because our space \mathcal{M}_0 is affine we can deduce by a Poincaré lemma style argument that α is exact. This direct proof is not appreciably shorter and we will discuss it elsewhere [CPRS2].

Corollary 7.9. Let (\mathcal{N}, D_0) be an odd unbounded θ -summable Breuer-Fredholm module for the Banach $*$ -algebra \mathcal{A} . Let $P = \chi_{[0,\infty)}(D_0)$, then for each unitary $u \in \mathcal{A}$ with $u(\text{dom } D_0) \subseteq \text{dom } D_0$ and $[D_0, u]$ bounded we have that PuP is a Breuer-Fredholm operator in PNP and if $\{D_t^u\}$ is any piecewise C^1 path in $\mathcal{M}_0 = D_0 + \mathcal{N}_{sa}$ from D_0 to uD_0u^* (for example the linear path) then

$$\text{ind}(PuP) = sf(\{D_t^u\}) = \frac{1}{\sqrt{\pi}} \int_0^1 \tau \left((D_t^u)' e^{-(D_t^u)^2} \right) dt.$$

Proof. See the proof of Corollary 7.5. \square

Corollary 7.10. Let (\mathcal{N}, D_0) be an odd unbounded θ -summable Breuer-Fredholm module for the Banach $*$ -algebra \mathcal{A} . For any $\epsilon > 0$ we define a one-form, α^ϵ on $\mathcal{M}_0 = D_0 + \mathcal{N}_{sa}$ by

$$\alpha^\epsilon(X) = \sqrt{\frac{\epsilon}{\pi}} \tau \left(X e^{-\epsilon D^2} \right)$$

for $D \in \mathcal{M}_0$ and $X \in T_D(\mathcal{M}_0) = \mathcal{N}_{sa}$. Then the integral of α^ϵ is independent of the piecewise C^1 path in \mathcal{M}_0 and if $\{D_t\}_{t \in [a,b]}$ is any piecewise C^1 path in \mathcal{M}_0 then

$$sf(D_a, D_b) = \sqrt{\frac{\epsilon}{\pi}} \int_a^b \tau \left(D'_t e^{-\epsilon D_t^2} \right) dt + \gamma_0(\sqrt{\epsilon} D_b) - \gamma_0(\sqrt{\epsilon} D_a).$$

Proof. Clearly, $(\mathcal{N}, \sqrt{\epsilon} D_0)$ is also a θ -summable module for \mathcal{A} and $sf(\sqrt{\epsilon} D_a, \sqrt{\epsilon} D_b) = sf(D_a, D_b)$ so that this is immediate from Theorem 7.8. \square

8. ETA INVARIANTS

Corollary 7.10 is very similar to Theorem 2.6 of [G] with some important differences. First, Getzler's theorem is for the type I_∞ case only; second, he assumes that the endpoints of the path, D_a and D_b are invertible; and third, his correction terms are truncated η -invariants. It is the purpose of this section to show that his correction terms are identical to ours when the endpoints are invertible, even in the type II_∞ setting. Moreover, when the endpoints are not invertible, we show how to modify the truncated η -invariants to get the right correction terms: namely, $\gamma_0(\sqrt{\epsilon} D)$.

First we define the truncated η -invariants and show that they make sense in our general setting.

Definition 8.1. *If D is an unbounded self-adjoint operator affiliated with \mathcal{N} , and e^{-tD^2} is trace-class for all $t > 0$ (briefly, D is θ -summable relative to \mathcal{N}) then we define*

$$\eta_\epsilon(D) = \frac{1}{\sqrt{\pi}} \int_\epsilon^\infty \tau \left(D e^{-tD^2} \right) t^{-1/2} dt.$$

We first observe that $D e^{-tD^2} = D e^{-(t/2)D^2} e^{-(t/2)D^2}$ where the first factor is a bounded operator (in \mathcal{N}) and the second factor is trace-class by hypothesis. Thus the integrand is finite-valued for each $t > 0$. Moreover by the functional calculus, the map $t \mapsto D e^{-(t/2)D^2}$ is operator norm continuous. As the second term equals $e^{t/2} e^{-(t/2)(1+D^2)}$ and $(1+D^2)^{-1} \in Li_0$ by Corollary B.6, we have that the second term is trace-class continuous by Corollary B.9. Thus the integrand is a continuous real-valued function. To see that the integral converges, we first prove the following.

Lemma 8.2. *Let D be an unbounded self-adjoint operator affiliated with \mathcal{N} such that $(1+D^2)^{-1} \in \mathcal{K}_{\mathcal{N}}$. Let $\{E_\lambda\}$ denote the spectral resolution of $|D|$ and suppose $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous and $f(|D|)$ is trace-class. Then,*

$$\tau(f(|D|)) = \int_0^\infty f(\lambda) d\phi_\lambda$$

where $\phi_\lambda = \tau(E_\lambda)$.

Proof. By spectral theory, $(1+D^2)^{-1} \geq \frac{1}{1+\lambda^2} E_\lambda$ so that ϕ_λ is a finite-valued increasing function. If we fix $\lambda_0 > 0$, then $f(|D|)E_{\lambda_0}$ lies in the II_1 algebra $E_{\lambda_0} \mathcal{N} E_{\lambda_0}$, on which τ is

operator-norm continuous. Then it is easy to see that

$$\tau(f(|D|)E_{\lambda_0}) = \tau\left(\int_0^{\lambda_0} f(\lambda)dE_\lambda\right) = \int_0^{\lambda_0} f(\lambda)d\phi_\lambda.$$

Now, as $\lambda_0 \rightarrow \infty$ the RHS approaches $\int_0^\infty f(\lambda)d\phi_\lambda$. By the lower semicontinuity of τ we get

$$\tau(f(|D|)) \leq \liminf_{\lambda_0 \rightarrow \infty} \tau(f(|D|)E_{\lambda_0}) \leq \limsup_{\lambda_0 \rightarrow \infty} \tau(f(|D|)E_{\lambda_0}) \leq \tau(f(|D|)),$$

and we are done. \square

A version of the following result is implicit in [M].

Lemma 8.3. *If D is θ -summable relative to \mathcal{N} , then the integral*

$$\int_1^\infty \tau(|D|e^{-tD^2})t^{-1/2}dt$$

converges.

Proof. We denote the spectral resolution of $|D|$ by $\{E_\lambda\}$. We let $\phi_\lambda = \tau(E_\lambda)$ then by the previous lemma,

$$\int_1^\infty \tau(|D|e^{-tD^2})t^{-1/2}dt = \int_1^\infty \int_0^\infty \lambda e^{-t\lambda^2} d\phi_\lambda t^{-1/2}dt.$$

To see that this double integral converges we first use Tonelli's theorem to interchange the order of integration

$$\int_0^\infty \int_1^\infty \lambda e^{-t\lambda^2} t^{-1/2} dt d\phi_\lambda = \int_0^\infty e^{-\lambda^2} \int_1^\infty \lambda e^{-(t-1)\lambda^2} t^{-1/2} dt d\phi_\lambda.$$

Make the substitution $v = (t-1)\lambda^2$ then we see that our double integral

$$\begin{aligned} &= \int_0^\infty \int_0^\infty e^{-v} (v + \lambda^2)^{-1/2} e^{-\lambda^2} dv d\phi_\lambda \\ &\leq \int_0^\infty e^{-v} v^{-1/2} dv \int_0^\infty e^{-\lambda^2} d\phi_\lambda \end{aligned}$$

which is finite as required since $\int_0^\infty e^{-\lambda^2} d\phi_\lambda = \tau(e^{-D^2})$. \square

Corollary 8.4. *If D is θ -summable relative to \mathcal{N} then for each $\epsilon > 0$, the integral*

$$\eta_\epsilon(D) = \frac{1}{\sqrt{\pi}} \int_\epsilon^\infty \tau(D e^{-tD^2}) t^{-1/2} dt$$

converges.

Proof. By replacing D with $\sqrt{\epsilon}D$, we can take $\epsilon = 1$. Convergence now follows from the previous lemma since $|\tau(D e^{-tD^2})| \leq \tau(|D|e^{-tD^2})$. \square

In order to reconcile $\eta_\epsilon(D)$ with the correction terms, $\gamma_0(\sqrt{\epsilon}D)$ of Corollary 7.10, we recall that

$$\gamma_0(\sqrt{\epsilon}D) = \gamma_{\frac{3}{2},1}(\sqrt{\epsilon}D(1 + \epsilon D^2)^{-1/2}).$$

Since $\sqrt{\epsilon}D(1 + \epsilon D^2)^{-1/2} = D(\frac{1}{\epsilon} + D^2)^{-1/2}$, we are led to consider the transformations

$$F_s = D(s + D^2)^{-1/2} \text{ for } s > 0.$$

Lemma 8.5. *Let D be an unbounded self-adjoint operator and let $r > 0$, $s > 0$. Then,*

- (1) $\|[(s + D^2)^{1/2} + (r + D^2)^{1/2}]^{-1}\| \leq \frac{1}{\sqrt{s} + \sqrt{r}}$, and
 (2) $\|[(s + D^2)^{1/2} + (r + D^2)^{1/2}]^{-1} - [2(s + D^2)^{1/2}]^{-1}\| \leq \frac{1}{\sqrt{s} + \sqrt{r}}|\sqrt{r} - \sqrt{s}|\frac{1}{2\sqrt{s}}.$

Proof. Item (1) follows from the functional calculus and the numerical inequality:

$$\frac{1}{\sqrt{s + x^2} + \sqrt{r + x^2}} \leq \frac{1}{\sqrt{s} + \sqrt{r}}$$

for all real x .

Item (2) follows from the numerical identity:

$$\frac{1}{\sqrt{s + x^2} + \sqrt{r + x^2}} - \frac{1}{2\sqrt{s + x^2}} = \left(\frac{\sqrt{s + x^2} - \sqrt{r + x^2}}{\sqrt{s + x^2} + \sqrt{r + x^2}} \right) \left(\frac{1}{2\sqrt{s + x^2}} \right)$$

and the easily proved estimate:

$$|\sqrt{s + x^2} - \sqrt{r + x^2}| \leq |\sqrt{s} - \sqrt{r}|.$$

□

Proposition 8.6. *If D is θ -summable relative to \mathcal{N} and $F_s = D(s + D^2)^{-1/2}$ for all $s > 0$ then $F_s \in F_D + (Li_0)_{F_D}$ and the mapping $s \mapsto F_s$ is C^1 in this space. Moreover, $\frac{d}{ds}(F_s) = -\frac{1}{2}F_s(s + D^2)^{-1}$.*

Proof. We first observe that for $s > 0$, $(s + D^2)^{-1} = (1/s)(1 + (1/s)D^2)^{-1}$ is in Li_0 by Corollary B.6. Now we fix $s > 0$. Then,

$$\begin{aligned} F_r - F_s &= D(r + D^2)^{-1/2} [(s + D^2)^{1/2} - (r + D^2)^{1/2}] (s + D^2)^{-1/2} \\ &= F_r [\{(s + D^2) - (r + D^2)\} \{(s + D^2)^{1/2} + (r + D^2)^{1/2}\}^{-1}] (s + D^2)^{-1/2} \\ &= F_r [(s - r) \{(s + D^2)^{1/2} + (r + D^2)^{1/2}\}^{-1}] (s + D^2)^{-1/2}. \end{aligned}$$

So, $F_s - F_r \in Li_0^{1/2}$ and in particular, $F_s - F_D \in Li_0^{1/2}$. Since $1 - F_s^2 = s(s + D^2)^{-1} \in Li_0$ we see that $F_s \in F_D + (Li_0)_{F_D}$ as claimed.

Now,

$$\frac{1}{r - s}(F_r - F_s) = -F_r [(s + D^2)^{1/2} + (r + D^2)^{1/2}]^{-1} (s + D^2)^{-1/2}$$

and so by the estimates of the previous lemma and the fact that $r \mapsto F_r$ is (at least!) operator norm continuous by the previous equations, we can take the limit as $r \rightarrow s$ in the norm of $Li_0^{1/2}$ to get:

$$\frac{d}{ds}(F_s) = -\frac{1}{2}F_s(s + D^2)^{-1} \text{ in the } Li_0^{1/2} \text{ sense.}$$

It is easily seen that $s \mapsto -(1/2)F_s(s + D^2)^{-1}$ is continuous in the norm of Li_0 and *a fortiori* in the norm of $(Li_0)_{F_D}$. To see that this derivative exists in the sense of the norm on $(Li_0)_{F_D}$, it suffices (by Lemma B.15) to see that

$$s \mapsto (1 - F_s^2) = s(s + D^2)^{-1} \in Li_0 \text{ is } C^1.$$

Clearly it suffices to see that $s \mapsto (s + D^2)^{-1}$ is C^1 in Li_0 . This is an easy resolvent equation calculation. In fact:

$$\frac{d}{ds}(s + D^2)^{-1} = -(s + D^2)^{-2}.$$

□

Lemma 8.7. *If D is θ -summable relative to \mathcal{N} and $\epsilon > 0$, then*

$$-\frac{1}{2}\eta_\epsilon(D) = \frac{1}{C_{\frac{3}{2},1}} \int_0^1 \tau \left(\frac{d}{ds}(F_s)(1 - F_s^2)^{-\frac{3}{2}} e^{-(1-F_s^2)^{-1}} \right) ds$$

where for $s \geq 0$, $F_s = \sqrt{\epsilon}D(s + \epsilon D^2)^{-1/2}$.

Proof. Since $\eta_\epsilon(D) = \eta_1(\sqrt{\epsilon}D)$, we can let $\epsilon = 1$ by replacing $\sqrt{\epsilon}D$ with D . Now,

$$\begin{aligned} -\frac{1}{2}\eta_1(D) &= -\frac{1}{2\sqrt{\pi}} \int_1^\infty \tau \left(t^{-1/2} D e^{-tD^2} \right) dt \\ &= -\frac{e}{\sqrt{\pi}} \int_1^\infty \tau \left(\frac{1}{2} t^{-1/2} D e^{-(1+tD^2)} \right) dt. \end{aligned}$$

Since this integral converges absolutely, we can make the (scalar) change of variable $t = 1/s$ to obtain the absolutely convergent integral:

$$\begin{aligned} -\frac{1}{2}\eta_1(D) &= -\frac{e}{\sqrt{\pi}} \int_0^1 \tau \left(\frac{1}{2} s^{-3/2} D e^{-(1+(1/s)D^2)} \right) ds \\ &= -\frac{e}{\sqrt{\pi}} \int_0^1 \tau \left(\frac{1}{2} s^{-3/2} D (1 + (1/s)D^2)^{-3/2} (1 + (1/s)D^2)^{3/2} e^{-(1+(1/s)D^2)} \right) ds \\ &= \frac{e}{\sqrt{\pi}} \int_0^1 \tau \left(\frac{d}{ds}(F_s)(1 - F_s^2)^{-3/2} e^{-(1-F_s^2)^{-1}} \right) ds \end{aligned}$$

by the previous proposition and the fact that $1 - F_s^2 = s(s + D^2)^{-1}$. We have previously noted that $C_{\frac{3}{2},1} = \frac{\sqrt{\pi}}{e}$ so we are done. □

Remarks. Because the path $F_s = D(s + D^2)^{-1/2}$ is not continuous at zero in the $(Li_0)_{F_D}$ -norm, in order to prove that this latter integral equals the integral of the one-form $\alpha_{\frac{3}{2}}$ along some (any) C^1 path from F_0 to $F_1 = F_D$ we cannot just appeal to the exactness of our one-form since we are integrating along a discontinuous path. To overcome this we argue as follows.

First we truncate our path at $\delta > 0$ where δ is small. Then we have $\{F_s\}$ for $\delta \leq s \leq 1$ is a C^1 path joining F_δ and $F_1 = F_D$. If we extend this path at its beginning with the straight line path from F_0 to F_δ , we obtain a piecewise C^1 path from F_0 to F_D . Thus the integral of our one-form along this new path is the same as the integral of our one-form along any C^1 path from F_0 to F_D . For small δ , the piece we have thrown away is small by the absolute

convergence of the integral. To complete the argument we must show that the piece we have added, namely the integral of our one-form along the straight line from F_0 to F_δ is also small. This is not obvious, since in the generic type II_∞ case, we would have $\|F_0 - F_\delta\| = 1$ for all $\delta > 0$, so that F_0 and F_δ would be even farther apart in $(Li_0)_{F_D}$ -norm.

Notation. For the purposes of the rest of this section we will use the notation $\theta(F_1, F_2)$ to denote the integral of the one-form $\alpha_{\frac{3}{2}}$ from F_1 to F_2 along a piecewise C^1 path in $F_D + (Li_0)_{F_D}$.

Lemma 8.8. *If D is θ -summable relative to \mathcal{N} and $0 < \delta < 1$, then*

$$\lim_{\delta \rightarrow 0} \theta(F_0, F_\delta) = 0,$$

where $F_\delta = D(\delta + D^2)^{-1/2}$.

Proof. Let $F_{\delta,t} = F_0 + t(F_\delta - F_0)$ for $t \in [0, 1]$ be the straight line path from F_0 to F_δ . Then

$$\theta(F_0, F_\delta) = \frac{1}{C_{\frac{3}{2},1}} \int_0^1 \tau \left((F_\delta - F_0)(1 - F_{\delta,t}^2)^{-3/2} e^{-(1-F_{\delta,t}^2)^{-1}} \right) dt.$$

We observe that the operator in the integrand is 0 on $\ker(D)$ because of the term $(F_\delta - F_0)$ and so all of the functions of D can be regarded as being restricted to $\ker(D)^\perp$. That is, for the purposes of this calculation, we can (and do) assume that $\ker(D) = \{0\}$. With this in mind, we factor the operator in the integrand into three pieces:

$$\begin{aligned} & (F_\delta - F_0), \\ & (1 - F_{\delta,t}^2)^{-3/2} e^{-\frac{1}{2}(1-F_{\delta,t}^2)^{-1}}, \text{ and} \\ & e^{-\frac{1}{2}(1-F_{\delta,t}^2)^{-1}}. \end{aligned}$$

The first factor is operator-norm bounded by 1. The second factor is operator-norm bounded (independent of t and δ) by:

$$\sup_{x \in [0,1]} \left[x^{-\frac{3}{2}} e^{-\frac{1}{2x}} \right] = \left(\frac{3}{e} \right)^{3/2} < 1.6.$$

The third factor is bounded as a positive operator by:

$$e^{-\frac{1}{2}(1-F_\delta^2)^{-1}}.$$

Thus, it suffices to see that

$$\|e^{-\frac{1}{2}(1-F_\delta^2)^{-1}}\|_1 = \tau \left(e^{-\frac{1}{2}(1-F_\delta^2)^{-1}} \right) \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

Now, for $0 < \delta \leq 1$,

$$1 - F_\delta^2 = \delta(\delta + D^2)^{-1} = \delta(1 + D^2)^{-1} [1 - (1 - \delta)(1 + D^2)^{-1}]^{-1},$$

and since $f(x) = \delta x[1 - (1 - \delta)x]^{-1}$ is an increasing function of x for $x \in [0, 1]$, we have by part (iv), Lemma 2.5 of [FK] that

$$\mu_t(1 - F_\delta^2) = \delta \mu_t((1 + D^2)^{-1}) [1 - (1 - \delta) \mu_t((1 + D^2)^{-1})]^{-1}.$$

Since we are assuming $\ker D = \{0\}$, we have for each fixed $t > 0$ that $0 \leq \mu_t((1 + D^2)^{-1})$ is strictly less than 1 and so:

$$\lim_{\delta \rightarrow 0} \mu_t(1 - F_\delta^2) = 0.$$

Another application of part (iv), Lemma 2.5 of [FK] gives us:

$$\lim_{\delta \rightarrow 0} \mu_t \left(e^{-\frac{1}{2}(1 - F_\delta^2)^{-1}} \right) = \lim_{\delta \rightarrow 0} \left(e^{-\frac{1}{2}[\mu_t(1 - F_\delta^2)]^{-1}} \right) = 0.$$

Therefore, by Corollary 2.8 of [FK] and the Lebesgue Dominated Convergence Theorem:

$$\tau \left(e^{-\frac{1}{2}(1 - F_\delta^2)^{-1}} \right) = \int_0^\infty e^{-\frac{1}{2}[\mu_t(1 - F_\delta^2)]^{-1}} dt \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

This completes the proof. \square

Theorem 8.9. *If D is θ -summable relative to \mathcal{N} and $\epsilon > 0$, then*

$$\frac{1}{2}\eta_\epsilon(D) = \gamma_0(\sqrt{\epsilon}D) - \frac{1}{2}\tau([ker(D)]),$$

where $[ker(D)]$ is the projection on $ker(D)$.

Proof. Since $\eta_\epsilon(D) = \eta_1(\sqrt{\epsilon}D)$, we can assume that $\epsilon = 1$. Combining Lemma 8.7, the Remarks, and Lemma 8.8 we now have:

$$\begin{aligned} \frac{1}{2}\eta_1(D) &= -\theta(F_0, F_D) \\ &= \theta(F_D, F_0) \\ &= \theta(F_D, \tilde{F}) - \theta(F_0, \tilde{F}) \\ &= \gamma_{\frac{3}{2},1}(F_D) - \gamma_{\frac{3}{2},q}(F_0) \\ &= \gamma_0(D) - \gamma_{\frac{3}{2},1}(F_0). \end{aligned}$$

Since $\tilde{F} - F_0 = [ker(D)]$, it is an easy calculation that:

$$\gamma_{\frac{3}{2},1}(F_0) = \frac{1}{2}\tau([ker(D)]),$$

and we're done. Another explanation of this last equality which does not directly involve calculating an integral is the following. Let $E = [ker(D)]$ and let $\hat{F} = \tilde{F} - 2E$. Then, \hat{F} and \tilde{F} are unitarily equivalent and clearly, $sf(\hat{F}, \tilde{F}) = \tau(E)$ is the integral of the one-form $\alpha_{\frac{3}{2}}$ from \hat{F} to \tilde{F} (by Theorem 5.8). Now, F_0 lies in a position of symmetry exactly half way between \hat{F} and \tilde{F} , and so the integrals from \hat{F} to F_0 and F_0 to \tilde{F} are identical with sum $\tau(E)$. Since $\gamma_{\frac{3}{2},1}(F_0)$ is by definition the integral from F_0 to \tilde{F} , we see that it is exactly $\frac{1}{2}\tau(E)$ as claimed. \square

♠ The η -invariant is focussed on the spectral asymmetry of the operator D , and so it treats 0 in a symmetric manner: $\eta_1(D)$ is an integral along a path connecting F_D to $sgn(D)$ where the *signum* function, sgn has the value 0 at 0. On the other hand, γ is concerned directly with spectral flow: it is the integral along a path from F_D to a “universal” symmetry (up to unitary equivalence) associated with D and so a natural choice is the *signum* function $sign$ which takes the value 1 at 0. ♠

Combining Corollary 7.10 and the previous theorem, we can now deduce the the promised generalization of Theorem 2.6 of [G]. We again observe that it applies equally to the type II_∞ situation and that we do not have to assume invertibility of the endpoints.

Corollary 8.10. *Let (\mathcal{N}, D_0) be an odd unbounded θ -summable Breuer-Fredholm module for the Banach $*$ -algebra \mathcal{A} . For any $\epsilon > 0$ we define a one-form, α^ϵ on $\mathcal{M}_0 = D_0 + \mathcal{N}_{sa}$ by*

$$\alpha^\epsilon(X) = \sqrt{\frac{\epsilon}{\pi}} \tau \left(X e^{-\epsilon D^2} \right)$$

for $D \in \mathcal{M}_0$ and $X \in T_D(\mathcal{M}_0) = \mathcal{N}_{sa}$. Then the integral of α^ϵ is independent of the piecewise C^1 path in \mathcal{M}_0 and if $\{D_t\}_{t \in [a,b]}$ is any piecewise C^1 path in \mathcal{M}_0 then

$$sf(D_a, D_b) = \sqrt{\frac{\epsilon}{\pi}} \int_a^b \tau \left(D'_t e^{-\epsilon D_t^2} \right) dt + \frac{1}{2} \eta_\epsilon(D_b) - \frac{1}{2} \eta_\epsilon(D_a) + \frac{1}{2} \tau([ker(D_b)] - [ker(D_a)]).$$

9. FINITELY SUMMABLE MODULES REVISITED

We show in this section how an application of Corollary 7.10 and the Laplace Transform combine to give a “best possible” version of Theorem 2.17 of [CP1]. The importance of this result is that we can use it in the case of $(1, \infty)$ -Breuer-Fredholm modules to obtain Connes’ Dixmier-trace formula for the index in a much wider setting [CPSu]. We do this by computing the limit as $p \rightarrow 1$ in the best possible formula.

Lemma 9.1. *If $n > 0$ (not necessarily an integer) and D is an unbounded self-adjoint operator affiliated with \mathcal{N} , such that $(1 + D^2)^{-n}$ is trace-class, then the integral $\int_0^\infty e^{-t(1+D^2)} t^{(n-1)} dt$ converges in both trace-norm and operator-norm to $\Gamma(n)(1 + D^2)^{-n}$.*

Proof. Since $(1 + D^2)^{-1}$ is in \mathcal{L}^n , $\mu_s((1 + D^2)^{-1})$ is $O(\frac{1}{s^{1/n}})$ by Lemma B.2 and so $(1 + D^2)^{-1} \in Li_0$. Now for $t > 0$, the map $t \mapsto \frac{1}{t}(1 + D^2)^{-1}$ is clearly continuous in Li_0 and hence $t \mapsto e^{-t(1+D^2)} \in \mathcal{L}^1$ is continuous for $t > 0$ by Corollary B.9. Thus the integrand is a continuous \mathcal{L}^1 -valued function of t . Now

$$\begin{aligned} \int_0^\infty \|e^{-t(1+D^2)} t^{(n-1)}\|_1 dt &= \int_0^\infty \tau(e^{-t(1+D^2)} t^{(n-1)}) dt \\ &= \int_0^\infty t^{(n-1)} \int_0^\infty e^{-\frac{t}{\mu_s((1+D^2)^{-1})}} ds dt \end{aligned}$$

by [FK] Corollary 2.8. As the integrand is positive we may interchange the order of integration by Tonelli’s theorem to obtain

$$\int_0^\infty \left(\int_0^\infty t^{(n-1)} e^{-\frac{t}{\mu_s((1+D^2)^{-1})}} dt \right) ds = \int_0^\infty \Gamma(n) \left(\frac{1}{\mu_s((1 + D^2)^{-1})} \right)^{-n} ds$$

by the Laplace Transform. Now, this equals

$$\int_0^\infty \Gamma(n) \mu_s((1 + D^2)^{-1})^n ds = \Gamma(n) \tau((1 + D^2)^{-n})$$

by [FK] Corollary 2.8. Thus, the integral converges in \mathcal{L}^1 - norm.

Similarly the integrand of the statement of the Lemma is operator-norm continuous and

$$\begin{aligned} \int_0^\infty \|e^{-t(1+D^2)} t^{(n-1)}\| dt &\leq \int_0^\infty t^{(n-1)} e^{-\frac{t}{\|(1+D^2)^{-1}\|}} dt \\ &= \Gamma(n) \|(1+D^2)^{-1}\|^n \end{aligned}$$

so the integral also converges in operator-norm. Clearly this limit operator is non-negative.

Let $\{E_\lambda\}_{\lambda \in [0,1]}$ be the spectral resolution of $(1+D^2)^{-1}$. Then for $\xi \in \mathcal{H}$ we have by the Spectral Theorem,

$$\begin{aligned} \langle \Gamma(n)(1+D^2)^{-n} \xi, \xi \rangle &= \int_0^1 \Gamma(n) \lambda^n d \langle E_\lambda \xi, \xi \rangle \\ &= \int_0^1 \left(\int_0^\infty t^{n-1} e^{-t/\lambda} dt \right) d \langle E_\lambda \xi, \xi \rangle \\ &= \int_0^\infty t^{n-1} \left(\int_0^1 e^{-t/\lambda} d \langle E_\lambda \xi, \xi \rangle \right) dt \quad \text{by Tonelli's Theorem} \\ &= \int_0^\infty t^{n-1} \langle e^{-t(1+D^2)} \xi, \xi \rangle dt = \langle \left(\int_0^\infty t^{n-1} e^{-t(1+D^2)} dt \right) \xi, \xi \rangle. \end{aligned}$$

Hence,

$$\Gamma(n)(1+D^2)^{-n} = \int_0^\infty t^{n-1} e^{-t(1+D^2)} dt$$

where the integral converges in both norms as claimed. \square

Lemma 9.2. *If $(1+D_0^2)^{-1}$ is in \mathcal{L}^n and $\{D_t\}$ is a piecewise C^1 path in $\mathcal{M}_0 = D_0 + \mathcal{N}_{sa}$ then*

$$\int_0^\infty \epsilon^{n-1} e^{-\epsilon} \int_0^1 \tau(D'_t e^{-\epsilon D_t^2}) dt d\epsilon$$

converges absolutely.

Proof.

$$\begin{aligned} \int_0^\infty \epsilon^{n-1} e^{-\epsilon} \int_0^1 |\tau(D'_t e^{-\epsilon D_t^2})| dt d\epsilon &\leq \sup_t \|D'_t\| \int_0^\infty \epsilon^{n-1} e^{-\epsilon} \int_0^1 \|e^{-\epsilon D_t^2}\|_1 dt d\epsilon \\ &= C \int_0^\infty \epsilon^{n-1} \int_0^1 \tau(e^{-\epsilon(1+D_t^2)}) dt d\epsilon \\ &= C \int_0^1 \int_0^\infty \tau(\epsilon^{n-1} e^{-\epsilon(1+D_t^2)}) d\epsilon dt \quad (\text{Tonelli}) \\ &= C \int_0^1 \tau(\Gamma(n)(1+D_t^2)^{-n}) dt \quad (\text{previous Lemma}) \\ &\leq C \Gamma(n) \sup_t f(\|D_t - D_0\|)^n \int_0^1 \tau((1+D_0^2)^{-n}) dt \end{aligned}$$

by Lemma 6 and Corollary 4 of appendix B of [CP1]. \square

Remarks. For unbounded p -summable modules we can now prove a “best possible” result, at least when the endpoints of the path are unitarily equivalent. The theorem below is optimal in two ways: first, the exponent $p/2$ is the minimum for which the formula makes sense, and second, we need no assumptions about the integrality of p or $p/2$. This result was conjectured in Appendix C of [CP1].

By similar methods, we can also derive an improved version of Theorem 2.16 of [CP1] when the endpoints are not unitarily equivalent. However, the exponent we need in this case, $(p+1)/2$, is not optimal. The reason for the extra $1/2$ in the exponent is that when we apply the Laplace Transform trick to the (truncated eta) correction terms, we need an exponent n which makes $D(1+D^2)^{-n}$ trace-class. When D is p -summable, the minimum such n is $(p+1)/2$. We omit this result, leaving the details to the interested reader.

Theorem 9.3 (cf Theorem 2.16 of [CP1]). *Let (\mathcal{N}, D_0) be an odd unbounded p -summable Breuer-Fredholm module (for \mathbf{C}) and let $\mathcal{M}_0 = D_0 + \mathcal{N}_{sa}$. Then for $D \in \mathcal{M}_0$, $X \in T_D(\mathcal{M}_0) = \mathcal{N}_{sa}$,*

$$X \mapsto \frac{1}{\tilde{C}_{p/2}} \tau(X(1+D^2)^{-p/2})$$

is an exact one-form on \mathcal{M}_0 . Moreover, if $\{D_t\}_{t \in [a,b]}$ is a piecewise C^1 path in \mathcal{M}_0 with D_a and D_b unitarily equivalent then

$$sf(D_a, D_b) = \frac{1}{\tilde{C}_{p/2}} \int_a^b \tau\left(\frac{d}{dt}(D_t)(1+D_t^2)^{-p/2}\right) dt,$$

where $\tilde{C}_{p/2} = \int_{-\infty}^{\infty} (1+x^2)^{-p/2} dx$.

Proof. By Corollary 7.10, we have for each $\epsilon > 0$:

$$sf(D_a, D_b) = \sqrt{\frac{\epsilon}{\pi}} \int_a^b \tau(D'_t e^{-\epsilon D_t^2}) dt.$$

Letting $n = p/2$, the Laplace Transform gives

$$1 = \frac{1}{\Gamma(n - \frac{1}{2})} \int_0^\infty \epsilon^{n-3/2} e^{-\epsilon} d\epsilon.$$

Thus combining these expressions yields

$$\begin{aligned} sf(D_a, D_b) &= \frac{1}{\Gamma(n - \frac{1}{2})} \int_0^\infty \epsilon^{n-3/2} e^{-\epsilon} \sqrt{\frac{\epsilon}{\pi}} \int_a^b \tau(D'_t e^{-\epsilon D_t^2}) dt d\epsilon \\ &= \frac{1}{\Gamma(n - \frac{1}{2}) \sqrt{\pi}} \int_a^b \int_0^\infty \epsilon^{n-1} \tau(D'_t e^{-\epsilon(1+D_t^2)}) d\epsilon dt \\ &= \frac{1}{\Gamma(n - \frac{1}{2}) \sqrt{\pi}} \int_a^b \int_0^\infty \tau[D'_t \epsilon^{n-1} e^{-\epsilon(1+D_t^2)}] d\epsilon dt \end{aligned}$$

using Lemma 9.2 and Fubini. Using Lemma 9.1 this double integral becomes

$$\begin{aligned} &= \frac{1}{\Gamma(n - \frac{1}{2})\sqrt{\pi}} \int_a^b \tau[D'_t \int_0^\infty \epsilon^{n-1} e^{-\epsilon(1+D_t^2)} d\epsilon] dt \\ &= \frac{\Gamma(n)}{\Gamma(n - \frac{1}{2})\sqrt{\pi}} \int_a^b \tau(D'_t(1 + D_t^2)^{-n}) dt. \end{aligned}$$

Finally, the normalization constant is the beta function:

$$B(n - 1/2, 1/2) = \frac{\Gamma(n - \frac{1}{2})\sqrt{\pi}}{\Gamma(n)} = \frac{\Gamma(n - \frac{1}{2})\Gamma(1/2)}{\Gamma(n)} = \int_0^1 t^{(n-3/2)}(1-t)^{-1/2} dt$$

by [Ru] Theorem 8.20. By the change of variables $t = 1/(1+x^2)$ this is $\int_{-\infty}^\infty (1+x^2)^{-n} dx$ which is the constant \tilde{C}_n .

It follows that $sf(D_a, D_b)$ is given by the integral of our one-form when the endpoints are unitarily equivalent. Thus, the integral of our one-form around any closed path is 0, and so the integral of the one-form is independent of path, in general. Thus, the one-form is exact by Lemma 7.3. \square

Corollary 9.4 (cf Theorem 2.17 of [CP1]). *Let (\mathcal{N}, D_0) be an odd p -summable Breuer-Fredholm module for the unital Banach $*$ -algebra \mathcal{A} , and let $P = \chi_{[0,\infty)}(D_0)$. Then for each $u \in U(\mathcal{A})$ with $u(\text{dom} D_0) \subseteq \text{dom}(D_0)$ and $[D_0, u]$ bounded, PuP is a Breuer-Fredholm operator in $P\mathcal{N}P$ and if $\{D_t^u\}$ is any piecewise C^1 path in $\mathcal{M}_0 = D_0 + \mathcal{N}_{sa}$ from D_0 to uD_0u^* (e.g., the linear path lies in \mathcal{M}_0), then:*

$$\text{ind}(PuP) = sf(\{D_t^u\}) = \frac{1}{\tilde{C}_{p/2}} \int_0^1 \tau \left(\frac{d}{dt} (D_t^u) (1 + (D_t^u)^2)^{-p/2} \right) dt,$$

the integral of the exact one-form, $\frac{1}{\tilde{C}_{p/2}} \tau(X(1+D^2)^{-p/2})$ along the path $\{D_t^u\}$.

10. SPECTRAL FLOW AND THE JLO COCYCLE

In this section we generalise the main theorem of [G] relating the spectral flow formula and the JLO cocycle to the case of spectral flow in semifinite von Neumann algebras. We adopt a more concrete functional analytic method than in [G] to avoid having to introduce more background material and also because there are additional subtleties in the type II setting. Throughout this section, (\mathcal{N}, D) is an odd unbounded θ -summable Breuer-Fredholm module for a Banach $*$ -algebra \mathcal{A} (contained in \mathcal{N}) and $u \in \mathcal{A}$ is a unitary operator leaving $\text{dom}(D)$ invariant and satisfying $[D, u]$ is bounded. There are three steps which we divide into three subsections.

10.1. The graded space. We form a new graded Hilbert space $\mathcal{K} = \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathcal{H}$. Introduce the Clifford algebra on \mathbf{C}^2 with generators:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let σ_0 denote the 2×2 identity matrix. Then the grading on \mathcal{K} is given by

$$\Gamma = \sigma_2 \otimes \sigma_3 \otimes I = \sigma_2 \otimes \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

where I is the identity operator in \mathcal{N} . Let $u \in \mathcal{A}$ be a unitary and introduce the following operators on \mathcal{K} , all of which commute with Γ , by

$$D_0 = \sigma_2 \otimes \sigma_0 \otimes D = \sigma_2 \otimes \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}, \quad q = \sigma_3 \otimes \begin{pmatrix} 0 & -iu^{-1} \\ iu & 0 \end{pmatrix},$$

$$D_r = (1-r)D_0 - rqD_0q, \quad D_{r,s} = D_r + sq, \quad r \in [0, 1], s \in [0, \infty).$$

Notice that if we define $D_r \equiv D_{r,0}$ then,

$$D_r = \sigma_2 \otimes \begin{pmatrix} D + ru^{-1}[D, u] & 0 \\ 0 & D + ru[D, u^{-1}] \end{pmatrix} = D_0 + r\sigma_2 \otimes \begin{pmatrix} u^{-1}[D, u] & 0 \\ 0 & u[D, u^{-1}] \end{pmatrix}.$$

So that

$$\dot{D}_r = \sigma_2 \otimes \begin{pmatrix} u^{-1}[D, u] & 0 \\ 0 & u[D, u^{-1}] \end{pmatrix}.$$

Introduce a graded trace: $Str(a) = \frac{1}{2\sqrt{\pi}}\text{tr}(\Gamma a)$ for a trace class and so for example

$$Str(\dot{D}_r e^{-D_r^2}) = \frac{1}{\sqrt{\pi}}\text{tr}\{u^{-1}[D, u]e^{-(D+ru^{-1}[D, u])^2} - u[D, u^{-1}]e^{-(D+ru[D, u^{-1}])^2}\}.$$

Next we calculate

$$D_{r,s}^2 = D_r^2 + s(1-2r)\sigma_1 \otimes \begin{pmatrix} 0 & [D, u^{-1}] \\ -[D, u] & 0 \end{pmatrix} + s^2$$

which depends on the relation

$$D_0q + qD_0 = \sigma_1 \otimes \begin{pmatrix} 0 & [D, u^{-1}] \\ -[D, u] & 0 \end{pmatrix}$$

The preceding relation explains in part the reason for introducing \mathcal{K} and the grading: it converts commutators to anticommutators, for example:

$$\begin{aligned} & \int_0^1 Str(qe^{-tD_0^2}(D_0q + qD_0)e^{-(1-t)D_0^2})dt \\ &= \frac{1}{2\sqrt{\pi}} \int_0^1 \text{tr}\{\sigma_2\sigma_3\sigma_1 \otimes \begin{pmatrix} iu^{-1}e^{-tD^2}[D, u]e^{-(1-t)D^2} & 0 \\ 0 & -iue^{-tD^2}[D, u^{-1}]e^{-(1-t)D^2} \end{pmatrix}\}dt \\ &= -\frac{1}{\sqrt{\pi}} \int_0^1 \text{tr}(u^{-1}e^{-tD^2}[D, u]e^{-(1-t)D^2} - ue^{-tD^2}[D, u^{-1}]e^{-(1-t)D^2}). \end{aligned}$$

10.2. Changing the path of integration. We now want to see what our spectral flow formula looks like on \mathcal{K} .

Lemma 10.1.

$$\int_0^1 \text{Str}(\dot{D}_r e^{-D_r^2}) dr = 2sf\{D, u^{-1}Du\}.$$

Proof.

$$D_r^2 = \sigma_0 \otimes \begin{pmatrix} (D + ru^{-1}[D, u])^2 & 0 \\ 0 & (D + ru[D, u^{-1}])^2 \end{pmatrix}.$$

so that

$$\begin{aligned} & \int_0^1 \text{Str}(\dot{D}_r e^{-D_r^2}) dr \\ &= \frac{1}{2\sqrt{\pi}} \int_0^1 \text{tr}\{\Gamma\sigma_2 \otimes \begin{pmatrix} u^{-1}[D, u]e^{-(D+ru^{-1}[D, u])^2} & 0 \\ 0 & u[D, u^{-1}]e^{-(D+ru[D, u^{-1}])^2} \end{pmatrix}\} dr \\ &= \frac{1}{\sqrt{\pi}} \int_0^1 \text{tr}\left\{\begin{pmatrix} u^{-1}[D, u]e^{-(D+ru^{-1}[D, u])^2} & 0 \\ 0 & -u[D, u^{-1}]e^{-(D+ru[D, u^{-1}])^2} \end{pmatrix}\right\} dr \\ &= \frac{1}{\sqrt{\pi}} \int_0^1 \text{tr}\{u^{-1}[D, u]e^{-(D+ru^{-1}[D, u])^2}\} dr - \frac{1}{\sqrt{\pi}} \int_0^1 \text{tr}\{u[D, u^{-1}]e^{-(D+ru[D, u^{-1}])^2}\} dr \end{aligned}$$

Now the first term is $sf\{D, u^{-1}Du\}$ and the second is $sf\{D, uDu^{-1}\}$. As

$$sf\{D, u^{-1}Du\} = sf\{uD u^{-1}, D\} = -sf\{D, u^{-1}Du\}$$

so that, as required,

$$\int_0^1 \text{Str}(\dot{D}_r e^{-D_r^2}) dr = 2sf\{D, u^{-1}Du\}.$$

□

The main idea of [G] is to change the path of integration used to compute the spectral flow. This is achieved with the help of the next result.

Lemma 10.2. *Consider the affine space Φ of perturbations of D_0 given by*

$$\{D_0 + X \mid X \in M_2 \otimes M_2 \otimes \mathcal{N} \text{ is self-adjoint and even (i.e., } \Gamma X = X\Gamma)\}$$

Then the map

$$X \rightarrow \text{Str}(X e^{-(D_0 + Y)^2})$$

is an exact one-form.

Proof. Closedness of this one-form is proved by using the methods of Appendix C and then exactness follows from a Poincaré lemma for the affine space Φ (see the remark after Theorem 7.8). □

Now consider the rectangle R in \mathbf{R}^2 given by $0 \leq r \leq 1, 0 \leq s \leq s_0$ for some s_0 . Using the previous exactness result we conclude that the integral:

$$\int_{\partial R} \text{Str}\left(\frac{dD_{r,s_0}}{dr} e^{-D_{r,s_0}^2}\right)$$

around the boundary ∂R is zero. So we can replace our original integral by a sum of three integrals and calculate the contribution of each to the spectral flow.

Lemma 10.3.

$$\lim_{s_0 \rightarrow \infty} \int_0^1 \text{Str}\left(\frac{dD_{r,s_0}}{dr} e^{-D_{r,s_0}^2}\right) dr = 0$$

Proof. First we observe that

$$\text{Str}\left(\frac{dD_{r,s_0}}{dr} e^{-D_{r,s_0}^2}\right) = \text{Str}(\dot{D}_r e^{-D_{r,s_0}^2}).$$

Next notice that $D_{1/2} = \frac{1}{2}(D_0 - qD_0q)$ so that $D_{1/2}$ anticommutes with q . Then $(D_{1/2} + sq)^2 = D_{1/2}^2 + s^2$ so that

$$\text{Str}(\dot{D}_{1/2} e^{-(D_{1/2} + sq)^2}) = e^{-s^2} \text{Str}(\dot{D}_{1/2} e^{-D_{1/2}^2})$$

which decays exponentially to zero as $s \rightarrow \infty$. Let $A_r = D_r - D_{1/2} = (\frac{1}{2} - r)(D_0 + qD_0q)$. Since

$$D_0 + qD_0q = -\dot{D}_r = \sigma_2 \otimes \begin{pmatrix} -u^{-1}[D, u] & 0 \\ 0 & -u[D, u^{-1}] \end{pmatrix},$$

we see that A_r is bounded by a constant independent of $r \in [0, 1]$. Using [CP1] Corollary 8 Appendix B we know there are constants C, C' depending only on $\|D_0 + qD_0q\|$ such that

$$\text{tr}(e^{-(D_r + sq)^2}) \leq C \text{tr}(e^{-C'(D_{1/2} + sq)^2}).$$

Thus

$$|\text{Str}(\dot{D}_r e^{-D_{r,s_0}^2})| \leq \|D_0 + qD_0q\| \text{tr}(e^{-(D_r + sq)^2}) \leq C \|D_0 + qD_0q\| \text{tr}(e^{-C'(D_{1/2} + sq)^2})$$

so that

$$\text{Str}\left(\frac{dD_{r,s_0}}{dr} e^{-D_{r,s_0}^2}\right)$$

decays exponentially to zero uniformly in r as $s_0 \rightarrow \infty$ proving the result. \square

Lemma 10.4.

$$\int_0^\infty \text{Str}\left(\frac{dD_{1,s}}{ds} e^{-D_{1,s}^2}\right) ds = - \int_0^\infty \text{Str}\left(\frac{dD_{0,s}}{ds} e^{-D_{0,s}^2}\right) ds$$

Proof. Note first that

$$D_{1,s} = -qD_0q + sq = -q(D_0 - sq)q$$

so that

$$\text{Str}\left(\frac{dD_{1,s}}{ds} e^{-D_{1,s}^2}\right) = \text{Str}(qe^{-(D_0 - sq)^2})$$

using invariance under conjugation by q . Now let $\rho = \sigma_2 \otimes \sigma_0 \otimes I$. Then using $\rho q \rho = -q$ and $\rho \Gamma = \Gamma \rho$ we have

$$2\sqrt{\pi} \text{Str}\left(\frac{dD_{1,s}}{ds} e^{-D_{1,s}^2}\right) = \text{tr}(\Gamma q e^{-(D_0 - sq)^2}) = \text{tr}(\rho^2 \Gamma q e^{-(D_0 - sq)^2})$$

$$\begin{aligned}
&= \text{tr}(\rho \Gamma q e^{-(D_0 - sq)^2}) \rho = \text{tr}(\rho \Gamma q \rho e^{-(D_0 + sq)^2}) \\
&= -\text{tr}(\rho \Gamma \rho q e^{-(D_0 + sq)^2}) = 2\sqrt{\pi} \text{Str}\left(\frac{dD_{0,s}}{ds} e^{-D_{0,s}^2}\right) \quad (*)
\end{aligned}$$

□

Combining the above results yields the key observation of this subsection:

Corollary 10.5.

$$sf\{D, u^{-1}Du\} = \int_0^\infty \text{Str}\left(\frac{dD_{0,s}}{ds} e^{-D_{0,s}^2}\right) ds \quad (**)$$

10.3. The Duhamel argument. Given the last corollary the essential observation in [G] is to use the Duhamel Principle to evaluate

$$\begin{aligned}
\int_0^\infty \text{Str}\left(\frac{dD_{0,s}}{ds} e^{-D_{0,s}^2}\right) ds &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \text{tr}(\Gamma q e^{-(D_0^2 + s(D_0 q + q D_0) + s^2)}) ds \\
&= \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-s^2} \text{tr}(\Gamma q e^{-(D_0^2 + s(D_0 q + q D_0))}) ds. \quad (***)
\end{aligned}$$

To use Duhamel (see [Y], p. 438) we write $[D_0, q]_+ = D_0 q + q D_0$ and then, checking (by use of the Spectral Theorem) that the formal derivatives actually exist in operator norm,

$$\begin{aligned}
e^{-(D_0^2 + s[D_0, q]_+)} - e^{-D_0^2} &= - \int_0^1 \frac{d}{dt} \left(e^{-tD_0^2} e^{-(1-t)(D_0^2 + s[D_0, q]_+)} \right) dt \\
&= -s \int_0^1 e^{-tD_0^2} [D_0, q]_+ e^{-(1-t)(D_0^2 + s[D_0, q]_+)} dt \\
&= -s \int_0^1 e^{-tD_0^2} [D_0, q]_+ \left(e^{-(1-t)D_0^2} - \int_0^{1-t} \frac{d}{dx} \left(e^{-x D_0^2} e^{-(1-t-x)(D_0^2 + s[D_0, q]_+)} \right) dx \right) dt \\
&= \dots \\
&= \sum_{k=1}^\infty (-s)^k \int_{\Delta_k} e^{-t_k D_0^2} [D_0, q]_+ e^{-t_{k-1} D_0^2} [D_0, q]_+ \dots [D_0, q]_+ e^{-t_0 D_0^2} dt_k \dots dt_1
\end{aligned}$$

Where Δ_n is the standard n -simplex in \mathbf{R}^{n+1} given by

$$\{(t_0, t_1, \dots, t_n) | t_j \in [0, 1], \sum_{j=0}^n t_j = 1\}.$$

Since $\text{vol}(\Delta_k) = 1/k!$, and the integrand over each Δ_k can be estimated in trace-norm (using the general Hölder inequality) by $\| [D_0, q]_+ \|^k \text{tr}(e^{-D_0^2})$, it is not hard to see that the series converges in trace-norm. Thus, we can substitute the resulting formula for $e^{-(D_0^2 + s[D_0, q]_+)}$ into the integral formula $(***)$ for $sf\{D, u^{-1}Du\}$. After a little manipulation we must evaluate the following.

Lemma 10.6. *With the above hypotheses and notation, we have two cases.*

If $n = 2k + 1$ is odd, then:

$$\int_{\Delta_n} \text{Str}\{q e^{-t_0 D_0^2} [D_0, q]_+ e^{-t_1 D_0^2} [D_0, q]_+ \dots [D_0, q]_+ e^{-t_n D_0^2}\} dt_1 \dots dt_n$$

$$\begin{aligned}
&= -\frac{1}{\sqrt{\pi}} \int_{\Delta_n} (-1)^k \text{tr}\{u^{-1}e^{-t_0 D^2} [D, u] e^{-t_1 D^2} [D, u^{-1}] \dots [D, u] e^{-t_n D^2}\} dt_1 \dots dt_n \\
&\quad + \frac{1}{\sqrt{\pi}} \int_{\Delta_n} (-1)^k \text{tr}\{ue^{-t_0 D^2} [D, u^{-1}] e^{-t_1 D^2} [D, u] \dots [D, u^{-1}] e^{-t_n D^2}\} dt_1 \dots dt_n.
\end{aligned}$$

If n is even, then

$$\int_{\Delta_n} \text{Str}\{qe^{-t_0 D_0^2} [D_0, q]_+ e^{-t_1 D_0^2} [D_0, q]_+ \dots [D_0, q]_+ e^{-t_n D_0^2}\} dt_1 \dots dt_n = 0.$$

Proof. For $n = 2k + 1$ odd, this is a straightforward calculation. For n even, we observe that the Γq contributes a first tensor factor of $i\sigma_1$; each of the n copies of $[D_0, q]_+$ contributes a first tensor factor of σ_1 ; and each of $n + 1$ exponential terms contributes a first tensor factor of σ_0 . This yields a first tensor factor of $i\sigma_1$ and so the trace is zero. \square

Lemma 10.7. *With the above hypotheses and notation,*

$$\begin{aligned}
&\sum_{k=0}^{\infty} k! \int_{\Delta_n} (-1)^k \text{tr}\{u^{-1}e^{-t_0 D^2} [D, u] e^{-t_1 D^2} [D, u^{-1}] e^{-t_2 D^2} \dots [D, u] e^{-t_{2k+1} D^2}\} dt_1 \dots dt_{2k+1} \\
&= -\sum_{k=0}^{\infty} k! \int_{\Delta_n} (-1)^k \text{tr}\{ue^{-t_0 D^2} [D, u^{-1}] e^{-t_1 D^2} [D, u] e^{-t_2 D^2} \dots [D, u^{-1}] e^{-t_{2k+1} D^2}\} dt_1 \dots dt_{2k+1}.
\end{aligned}$$

Proof. This result is a simple consequence of the fact that the JLO formula defines a cocycle in the (b, B) -bicomplex (Theorem IV.21 in [Co4]: see also [JLO, GZ]). It is well known to experts but we could not find a good exposition in the literature and so for completeness we indicate the proof. The JLO cocycle is the sequence of multilinear functionals $(\phi_n) = \phi_{JLO}$ on (\mathcal{A}^{n+1}) , $n = 0, 1, 2, \dots$ where for $n = 2k + 1$ (this is the only case we need):

$$\begin{aligned}
&\phi_{2k+1}(a^0, a^1, \dots, a^{2k+1}) \\
&= \sqrt{2i} \int_{\Delta_n} (-1)^k \text{tr}\{a^0 e^{-t_0 D^2} [D, a^1] e^{-t_1 D^2} [D, a^2] e^{-t_2 D^2} \dots [D, a^{2k+1}] e^{-t_{2k+1} D^2}\} dt_1 \dots dt_{2k+1}.
\end{aligned}$$

Introduce the sequence for $k = 1, 2, \dots$

$$(1, u^{-1}, u, \dots, u^{-1}, u)_{2k+2} \in \mathcal{A}^{2k+3}$$

where 1 is the identity of \mathcal{A} and the subscript indicates the number of terms to the right of the first element in each term of the sequence. We calculate, using the standard formula for the Hochschild coboundary operator b [Co4]

$$b\phi_{2k+1}(1, u^{-1}, u, \dots, u^{-1}, u) = \phi_{2k+1}(u^{-1}, u, \dots, u^{-1}, u) + \phi_{2k+1}(u, u^{-1}, u, \dots, u^{-1}).$$

For the operator B of the (b, B) -bicomplex we use the formula $B = AB_0$ [Co4] where A is the antisymmetrisation operator and B_0 acts on an $n + 2$ linear functional ψ by

$$B_0\psi(a^0, a^1, \dots, a^n) = \psi(1, a^0, a^1, \dots, a^n) - (-1)^{n+1}\psi(a^0, a^1, \dots, a^n, 1).$$

Thus,

$$\begin{aligned}
B_0\phi_{2k+1}(1, u^{-1}, u, \dots, u^{-1}, u) &= \\
\phi_{2k+1}(1, 1, u^{-1}, u, \dots, u^{-1}, u) &- \phi_{2k+1}(1, u^{-1}, u, \dots, u^{-1}, u, 1) = 0
\end{aligned}$$

because the commutator with D in the JLO formula kills all terms with two copies of the identity operator. It follows that

$$B\phi_{2k+1}(1, u^{-1}, u, \dots, u^{-1}, u) = 0,$$

and since in the (b, B) -bicomplex we have that $(b + B)\phi_{2k+1} = 0$, we get

$$\begin{aligned} 0 &= b\phi_{2k+1}(1, u^{-1}, u, \dots, u^{-1}, u) \\ &= \phi_{2k+1}(u^{-1}, u, \dots, u^{-1}, u) + \phi_{2k+1}(u, u^{-1}, u, \dots, u^{-1}) \end{aligned}$$

which proves the statement of the lemma. \square

Hence we can use the preceding two lemmas to obtain our final formula:

$$\begin{aligned} &\int_0^\infty \text{Str}\left(\frac{dD_{0,s}}{ds} e^{-D_{0,s}^2}\right) ds = \\ &- \int_0^\infty \int_{\Delta_n} \sum_{n \text{ odd}} s^n e^{-s^2} \text{Str}\{q e^{-t_n D_0^2} [D_0, q]_+ e^{-t_{n-1} D_0^2} [D_0, q]_+ \dots [D_0, q]_+ e^{-t_0 D_0^2}\} dt_n \dots dt_1 ds \\ &= - \sum_{k=0}^\infty \frac{k!}{2} \int_{\Delta_{2k+1}} \text{Str}\{q e^{-t_{2k+1} D_0^2} [D_0, q]_+ e^{-t_{2k} D_0^2} [D_0, q]_+ \dots [D_0, q]_+ e^{-t_0 D_0^2}\} dt_{2k+1} \dots dt_1 \\ &= \frac{1}{\sqrt{\pi}} \sum_{k=0}^\infty (-1)^k k! \int_{\Delta_{2k+1}} \text{tr}\{u^{-1} e^{-t_0 D^2} [D, u] e^{-t_1 D^2} [D, u^{-1}] \dots [D, u] e^{-t_{2k+1} D^2}\} dt_{2k+1} \dots dt_1. \end{aligned}$$

Now we have, by Corollary 10.5 the connection between cyclic cohomology in the form of the JLO cocycle and spectral flow.

Theorem 10.8. *Let (\mathcal{N}, D) be an odd unbounded θ -summable Breuer-Fredholm module for the Banach $*$ -algebra \mathcal{A} and let $u \in \mathcal{A}$ be a unitary operator leaving $\text{dom}(D)$ invariant and satisfying $[D, u]$ is bounded. Then,*

$$\begin{aligned} &sf\{D, u^{-1}Du\} \\ &= \frac{1}{\sqrt{\pi}} \sum_{k=0}^\infty (-1)^k k! \int_{\Delta_{2k+1}} \text{tr}\{u^{-1} e^{-t_0 D^2} [D, u] e^{-t_1 D^2} [D, u^{-1}] \dots [D, u] e^{-t_{2k+1} D^2}\} dt_{2k+1} \dots dt_1. \end{aligned}$$

One may interpret this result as a Chern character formula pairing the $K_1(\mathcal{A})$ class $[u]$ of $u \in \mathcal{A}$ with an explicit entire cyclic cocycle for \mathcal{A} obtained via the semifinite K -homology class defined by D . We will not pursue this point of view here mentioning only the fact that the main interest in this result is that it allows us to prove an analogue of the Connes-Moscovici local index theorem [CoMo] in the semi-finite case.

APPENDIX A. OPERATOR IDEALS

This appendix establishes some properties of certain ideals of operators in a von Neumann algebra \mathcal{N} with a faithful, normal semifinite trace τ . A similar discussion is contained in section 5 of [Suk]. One can prove versions of the results in this section for the Marcinkiewicz spaces defined in section 5 of [Suk] (see also [DD]). Our point of view is slightly different from and more naïve than his: we try to establish the basic properties of these normed ideals with as little machinery as possible in hopes of making the material more accessible to those as naïve as ourselves. We claim no originality for the results themselves. We first recall Definition 2.1.

Definition A.1. *If $S \in \mathcal{N}$ the t -th generalized singular value of S for each real $t > 0$ is given by*

$$\mu_t(S) = \inf\{\|SE\| \mid E \text{ is a projection in } \mathcal{N} \text{ with } \tau(1 - E) \leq t\}.$$

For the basic properties of these singular values we refer to [FK].

Definition A.2. *If \mathcal{I} is a $*$ -ideal in \mathcal{N} which is complete in a norm $\|\cdot\|_{\mathcal{I}}$ then we will call \mathcal{I} an **invariant operator ideal** if*

- (1) $\|S\|_{\mathcal{I}} \geq \|S\|$ for all $S \in \mathcal{I}$,
- (2) $\|S^*\|_{\mathcal{I}} = \|S\|_{\mathcal{I}}$ for all $S \in \mathcal{I}$,
- (3) $\|ASB\|_{\mathcal{I}} \leq \|A\| \|S\|_{\mathcal{I}} \|B\|$ for all $S \in \mathcal{I}$, $A, B \in \mathcal{N}$.

Since \mathcal{I} is an ideal in a von Neumann algebra, it follows from I.1.6, Proposition 10 of [Dix] that if $0 \leq S \leq T$ and $T \in \mathcal{I}$, then $S \in \mathcal{I}$ and $\|S\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}}$. Much more is true, especially in the type I case but we shall not need it here, see [GK].

Examples.

- (1) Let $\mathcal{I} = Li = \{T \in \mathcal{N} \mid \mu_s(T) = O(1/\log s)\}$. The norm on Li is:

$$\|T\|_{Li} = \sup_{r>0} \left\{ \frac{\int_0^r \mu_s(T) ds}{\int_0^r (\log(s+e))^{-1} ds} \right\}.$$

We observe that $\|T\|_{Li} \geq \|T\|$.

- (2) We let $\mathcal{I} = Li_0 = \{T \in \mathcal{N} \mid \mu_s(T) = o(1/\log s)\}$ with the norm inherited from Li . Using the estimate

$$r(\log(r+e))^{-1} \leq \int_0^r (\log(s+e))^{-1} ds \leq 3r(2\log(r+e))^{-1}$$

of Lemma A.4 below it follows that Li_0 is the closure of the ideal \mathcal{F}_N of ‘finite rank’ operators in the Li norm, see [GK]. Here

$$\mathcal{F}_N = \{T \in \mathcal{N} \mid T = ET \text{ for some projection } E \in \mathcal{N} \text{ with } \tau(E) < \infty\}.$$

- (3) For $0 < q \leq 1$, let $\mathcal{I} = Li^q = \{T \in \mathcal{N} \mid \mu_s(T) = O((1/\log s)^q)\}$
 $= \{T \in \mathcal{N} \mid |T|^{1/q} \in Li\}.$

We use the following norm on Li^q :

$$\|T\|_{Li^q} = (\| |T|^{1/q} \|_{Li})^q.$$

We prove below that this is in fact a norm on Li^q and that Hölder's inequality is satisfied for these spaces. There is an equivalent norm on Li^q in which it is complete [Suk], given by:

$$\sup_{r>0} \left\{ \frac{\int_0^r \mu_s(T) ds}{\int_0^r (\log(s+e))^{-q} ds} \right\}.$$

We will not use this norm explicitly, however.

Lemma A.3. (1) For each q with $0 < q \leq 1$, we have $\|\cdot\|_{Li^q}$ is an invariant norm on Li^q .
(2) If $0 < q, q' \leq 1$ where $q + q' \leq 1$ and if $S \in Li^q$ and $T \in Li^{q'}$ then $ST \in Li^{q+q'}$ and

$$\|ST\|_{Li^{q+q'}} \leq \|S\|_{Li^q} \|T\|_{Li^{q'}}.$$

Proof. (1) The only nontrivial part is subadditivity. So, suppose $T, S \in Li^q$. Then,

$$\begin{aligned} \|T + S\|_{Li^q} &= \left(\| |T + S|^{1/q} \|_{Li} \right)^q \\ &= \sup_{t>0} \left[\frac{\int_0^t \mu_s(|T + S|^{1/q}) ds}{\int_0^t \frac{1}{\log(s+e)} ds} \right]^q \\ &\leq \sup_{t>0} \left[\frac{\int_0^t [\mu_s(T) + \mu_s(S)]^{1/q} ds}{\int_0^t \frac{1}{\log(s+e)} ds} \right]^q \quad (\text{Theorem 4.4 part (iii), [FK]}) \\ &\leq \sup_{t>0} \frac{\left[\int_0^t (\mu_s(T))^{1/q} ds \right]^q + \left[\int_0^t (\mu_s(S))^{1/q} ds \right]^q}{\left[\int_0^t \frac{1}{\log(s+e)} ds \right]^q} \\ &\leq \|T\|_{Li^q} + \|S\|_{Li^q}. \end{aligned}$$

(2) This proof is very similar to part (1) except we cite Theorem 4.2 part (iii) of [FK] and then apply the usual Hölder inequality for the interval $[0, t]$. \square

Lemma A.4. For $r \geq e$,

$$\int_e^r \frac{dx}{\log x} \leq \frac{3(r-e)}{2 \log r}$$

Proof. A straightforward calculus exercise. \square

Note that we may reformulate this inequality as

$$\int_0^r \frac{dx}{\log(x+e)} \leq \frac{3r}{2 \log(r+e)}.$$

Lemma A.5. (1) For $S \in Li^q$, $0 < q \leq 1$ let $f_q(t) = \mu_t(S)(\log(t+e))^q$ then

$$(2/3)^q \|f_q\|_\infty \leq \|S\|_{Li^q} \leq \|f_q\|_\infty.$$

(2) For $0 < q \leq 1$ the embedding $Li \hookrightarrow Li^q$ is bounded (by $3/2$) and, in fact, for $0 < q < 1$, we have $Li \subseteq Li_0^q$.

Proof. (1) On the one hand,

$$\begin{aligned} \|S\|_{Li^q} &= \sup_{r>0} \left[\frac{\int_0^r (\mu_t(S))^{1/q} dt}{\int_0^r \frac{1}{\log(t+e)} dt} \right]^q \\ &= \sup_{r>0} \left[\frac{\int_0^r \frac{(f_q(t))^{1/q}}{\log(t+e)} dt}{\int_0^r \frac{1}{\log(t+e)} dt} \right]^q \leq \|f_q\|_\infty. \end{aligned}$$

On the other hand by the previous lemma,

$$\begin{aligned} \|S\|_{Li^q} &\geq \sup_{r>0} \left[\frac{r\mu_r(S)^{1/q}}{1.5r/\log(r+e)} \right]^q \\ &= (2/3)^q \sup_{r>0} [\mu_r(S)(\log(r+e))^q] \\ &= (2/3)^q \|f_q\|_\infty. \end{aligned}$$

(2) By part (1),

$$\begin{aligned} \|S\|_{Li^q} \leq \|f_q\|_\infty &= \sup_{t>0} [\mu_t(S) \log(t+e)^q] \\ &\leq \sup_{t>0} [\mu_t(S) \log(t+e)] \\ &= \|f_1\|_\infty \leq (3/2) \|S\|_{Li}. \end{aligned}$$

If $0 < q < 1$, and $S \in Li$ so that $\mu_t(S) = O(1/\log t)$ then clearly $\mu_t(S) = o(1/(\log t)^q)$ so that $S \in Li_0^q$. \square

APPENDIX B. TRACE-CLASS CONTINUITY OF CERTAIN MAPS

The following result is well-known in the case of type I_∞ von Neumann algebras. Fyodor Sukochev pointed out to us that, in general, it follows from a result of [FK]. Moreover, the result has been extended by him and his co-authors to many symmetric operator spaces including the spaces Li^q , [CDS].

Proposition B.1. . *Let $\{T_n\}_{n=1}^\infty$ and T be positive trace class operators in \mathcal{N} . If both $\|T_n - T\| \rightarrow 0$ and $\|T_n\|_1 \rightarrow \|T\|_1$ then $\|T_n - T\|_1 \rightarrow 0$.*

Proof. It follows from $\|T_n - T\| \rightarrow 0$ that T_n converges to T in the measure topology, and so the result follows from [FK] Theorem 3.7. \square

The following two results were first pointed out to us by Chris Bose. They are well-known as the containment of L^p in $L^{p,\infty}$. We include the simple proofs of these known results for completeness, see [Suk].

Lemma B.2. *Let f be a non-negative decreasing function on \mathbb{R}^+ and suppose $f \in L^p$ for some $p > 0$ then $f(x) \leq \frac{\|f\|_p}{x^{1/p}}$ for all $x > 0$. In other words f is $O(\frac{1}{x^{1/p}})$ as $x \rightarrow \infty$.*

Proof. As f is decreasing, $f(x)\chi_{[0,x]} \leq f$ for each $x > 0$ and so

$$f(x)^p x = \int_0^\infty f(x)^p \chi_{[0,x]} du \leq \int_0^\infty f(u)^p du = \|f\|_p^p.$$

Hence $f(x) \leq \frac{\|f\|_p}{x^{1/p}}$. □

Corollary B.3. *If $f \in L^p(\mathbb{R}^+)$ satisfies the hypotheses of the lemma then f is $o(\frac{1}{x^{1/p}})$ as $x \rightarrow \infty$.*

Proof. Given $\epsilon > 0$, there is an x_0 such that $\int_{x_0}^\infty f^p \leq \epsilon^p/2$. Then for $x > x_0$:

$$f(x)^p(x - x_0) \leq \int_{x_0}^\infty f^p \leq \frac{\epsilon^p}{2}$$

or

$$f(x)^p \leq \frac{\epsilon^p}{2(x - x_0)}$$

implying $f(x)^p \leq \frac{\epsilon^p}{x}$ if $x \geq 2x_0$. That is, $f(x) \leq \frac{\epsilon}{x^{1/p}}$ if $x \geq 2x_0$. □

Lemma B.4. *An operator X is in Li if and only if there is a $t_0 > 0$ such that*

$$\tau(e^{-t|X|^{-1}}) < \infty$$

for all $t > t_0$.

Proof. By definition,

$$X \in Li \iff \mu_x(|X|) = O\left(\frac{1}{\log x}\right).$$

Thus $X \in Li$ is equivalent to the existence of t_0 such that for all $x \geq x_0$, $\mu_x(|X|) \leq \frac{t_0}{\log x}$ which in turn is equivalent to $e^{\frac{-t_0}{\mu_x(|X|)}} \leq 1/x$ for all $x \geq x_0$. This is equivalent to

$$e^{-\frac{t}{\mu_x(|X|)}} \leq \frac{1}{x^{t/t_0}}$$

for all $x \geq x_0$, $t > t_0$. Applying lemma 2.5 of [FK] now gives $X \in Li \iff$ there is a t_0 such that for all $t > t_0$, $\mu_x(e^{-t|X|^{-1}}) \leq \frac{1}{x^{t/t_0}}$ for all $x \geq x_0$. Hence the forward implication of the lemma follows by lemma 2.7 of [FK] as there is a t_0 such that for all $t > t_0$

$$\tau(e^{-t|X|^{-1}}) = \int_0^\infty \mu_x(e^{-t|X|^{-1}}) dx \leq C + \int_{x_0}^\infty \frac{1}{x^{t/t_0}} dx < \infty.$$

The reverse implication follows because

$$\tau(e^{-t_1|X|^{-1}}) = \int_0^\infty \mu_x(e^{-t_1|X|^{-1}}) dx < \infty$$

for some $t_1 > 0$ implies, by Corollary B.3 for $p = 1$ that $\mu_x(e^{-t_1|X|^{-1}})$ is $o(\frac{1}{x})$. So there is an $x_0 \geq 0$ so that $x \geq x_0$ implies $\mu_x(e^{-t_1|X|^{-1}}) \leq \frac{1}{x}$. But then $\mu_x(|X|) = O(\frac{1}{\log x})$ which implies that $X \in Li$. □

Remarks. By the first half of the proof, we can choose for t_0 any positive real number for which eventually in x , $\mu_x(|X|) \leq \frac{t_0}{\log x}$. So, if X is in Li and $\|X\|_{Li} < K$ then by Lemma A.5 we have:

$$\mu_t(X) \log(t) \leq \mu_t(X) \log(t + e) \leq (1.5)\|X\|_{Li} < (1.5)K,$$

and so $\tau(e^{-(1.5)K|X|^{-1}}) < \infty$.

Corollary B.5. *We have $X \in Li_0$ if and only if $\tau(e^{-t|X|^{-1}}) < \infty$ for all $t > 0$.*

Corollary B.6. *Let D be an unbounded self-adjoint operator affiliated with \mathcal{N} . Then*

- (i) $\tau(e^{-tD^2}) < \infty$ for all $t > 0$ iff $(1 + D^2)^{-1} \in Li_0$ (i.e., (\mathcal{N}, D) is θ -summable).
- (ii) $\tau(e^{-tD^2}) < \infty$ for all $t > t_0 > 0$ if and only if $(1 + D^2)^{-1} \in Li$ (i.e., (\mathcal{N}, D) is weakly θ -summable).

Lemma B.7. *If $S, T \in Li^q$ then*

$$\| |S| - |T| \|_{Li^q} \leq \sqrt{2} \|S - T\|_{Li^q}^{1/2} \|S + T\|_{Li^q}^{1/2}.$$

Proof. This is a special case of Theorem 2.1 of [DD] as Li^q is fully symmetric by [Suk]. \square

Remarks. Fyodor Sukochev has pointed out to us that the following theorem can be proved in much greater generality using results of O. E. Tikhonov on the continuity of operator functions in the measure topology combined with some results of V. Chilin and F. Sukochev on weak convergence in operator spaces. Using these techniques one is reduced to proving (in the case of the following theorem) that

$$\int_0^\infty f(\mu_t(T_n)) dt \rightarrow \int_0^\infty f(\mu_t(T)) dt.$$

Since this is the main difficult point in our proof, we prefer not to confuse the issue with unnecessary generality. However, his comments have led to a streamlining of our proof for which we are very grateful.

Theorem B.8. *Let \mathcal{I} be an invariant operator ideal in \mathcal{N} and let f be a continuous increasing function from \mathbb{R}^+ to itself such that $f(T)$ is trace-class for each $T \in \mathcal{I}_+$. Then, $T \mapsto f(T)$ mapping $\mathcal{I}_+ \rightarrow L^1$ is continuous.*

Proof. Suppose $\|T_n - T\|_{\mathcal{I}} \rightarrow 0$ in \mathcal{I}_+ . Then $S_n = T_n - T$ is self-adjoint and

$$0 \leq T_n = T + S_n \leq T + |S_n|$$

where $|S_n| \geq 0$ is also in \mathcal{I} with

$$\| |S_n| \|_{\mathcal{I}} = \|S_n\|_{\mathcal{I}} \rightarrow 0.$$

Now, since $\|\cdot\|_{\mathcal{I}} \geq \|\cdot\|$ we have $\|S_n\| \rightarrow 0$. Next, we note that by [FK] (Lemma 2.5 (i), (v)) we have for all $t > 0$

$$\begin{aligned} \mu_t(T_n) &= \mu_t(T + S_n) \leq \mu_t(T) + \|S_n\| \text{ and} \\ \mu_t(T) &= \mu_t(T_n - S_n) \leq \mu_t(T_n) + \|S_n\| \end{aligned}$$

Thus, $\mu_t(T_n) \rightarrow \mu_t(T)$ uniformly in t . Since the T_n are uniformly bounded in operator norm by, say C , and since f is uniformly continuous on $[0, C]$, we see that $f(\mu_t(T_n)) \rightarrow f(\mu_t(T))$ uniformly in t . We claim that

$$\tau(f(T_n)) = \int_0^\infty f(\mu_t(T_n)) dt \rightarrow \int_0^\infty f(\mu_t(T)) dt = \tau(f(T)).$$

The two equalities follow by [FK] (Corollary 2.8). The convergence

$$\tau(f(T_n)) \rightarrow \tau(f(T))$$

is a little subtle because we have a non-finite measure. It suffices to show that every subsequence has itself a subsequence which converges to $\tau(f(T))$. So given a subsequence $\{\tau(f(T_{n_k}))\}$ choose a further subsequence $T_{n_{k_j}} = T + S_{n_{k_j}} := T + R_j$ with the property that $\sum_{j=1}^\infty \|R_j\|_{\mathcal{I}} < \infty$. Then $\sum_{j=1}^\infty |R_j|$ converges in \mathcal{I}_+ to say $R \geq 0$. So we have $T + R \geq T + R_j$ for all j and also $T + R \geq T$ so that by [FK] (Lemma 2.5(iii)):

$$f(\mu_t(T + R)) \geq f(\mu_t(T_{n_{k_j}})) \text{ and } f(\mu_t(T + R)) \geq f(\mu_t(T)).$$

Now

$$\int_0^\infty f(\mu_t(T + R)) dt = \tau(f(T + R)) < \infty$$

since $T + R \in \mathcal{I}$. By dominated convergence $\tau(f(T_{n_{k_j}})) \rightarrow \tau(f(T))$ and hence $\tau(f(T_n)) \rightarrow \tau(f(T))$. By the functional calculus,

$$\|f(T_n) - f(T)\| \rightarrow 0$$

as f is bounded and continuous on $[0, C]$ so that using

$$\|f(T_n)\|_1 = \tau(f(T_n)) \rightarrow \tau(f(T)) = \|f(T)\|_1$$

we obtain $\|f(T_n) - f(T)\|_1 \rightarrow 0$ by Proposition B.1 of Appendix B. \square

Corollary B.9. *For any $b > 0$, the map $T \mapsto e^{-b|T|^{-1/q}}$ from Li_0^q to the trace-class operators is continuous.*

Proof. Since $T \mapsto |T|$ is continuous by Lemma B.7 we can assume that $T \geq 0$. The result follows from Corollary B.5 and Theorem B.8. \square

Corollary B.10. *For any $c \geq 0$, $b > 0$ the map $T \mapsto |T|^{-c} e^{-b|T|^{-1/q}}$ from Li_0^q to the trace-class operators is continuous.*

Proof. As in the previous proof we can assume that $T \geq 0$. Then,

$$T^{-c} e^{-bT^{-1/q}} = T^{-c} e^{-(b/2)T^{-1/q}} e^{-(b/2)T^{-1/q}}.$$

For $t \geq 0$, $t \mapsto t^{-c} e^{-(b/2)t^{-1/q}}$ is bounded and continuous with the understanding that it is zero at $t = 0$. So, $T \mapsto T^{-c} e^{-(b/2)T^{-1/q}}$ is operator-norm to operator-norm continuous $Li_0^q \rightarrow \mathcal{N}$. Since the other factor is continuous $Li_0^q \rightarrow L^1$ by the previous corollary and since the Li_0^q norm dominates the operator norm, the product is also continuous $Li_0^q \rightarrow L^1$. \square

Corollary B.11. *If $c \geq 0$ and $\epsilon > 0$, $b > 0$ then $T \mapsto |T|^{-c} e^{-b|T|^{-(1+\epsilon)}}$ from Li to the trace-class operators is continuous.*

Proof. Let $q = 1/(1+\epsilon)$. It is clear that $Li \subset Li_0^q$ and the inclusion is continuous by Lemma A.5. The result then follows from the preceding corollary. \square

Remarks. If $T \in Li$ and $\|T\|_{Li} < \frac{2}{3}$ then by the Remarks after Lemma B.4, $\tau(e^{-|T|^{-1}}) < \infty$. The proof of Theorem B.8 now shows that the map $T \mapsto e^{-|T|^{-1}}$ from $\{T \in Li \mid \|T\|_{Li} < \frac{2}{3}\}$ to L^1 is continuous. More generally, for any $C > 0$, the map $T \mapsto e^{-C|T|^{-1}}$ is continuous from $\{T \in Li \mid \|T\|_{Li} < \frac{2}{3}C\}$ to L^1 .

Now we choose a fixed self-adjoint Breuer-Fredholm operator $F_0 \in \mathcal{N}$ with $1 - F_0^2 \in \mathcal{I}$ for some invariant operator ideal \mathcal{I} , and recall the definition of Section 2:

$$\mathcal{I}_{F_0} = \{X \in \mathcal{I}_{sa}^{1/2} \mid 1 - (F_0 + X)^2 \in \mathcal{I}\}$$

Lemma B.12. *Suppose that \mathcal{I} and $\mathcal{I}^{1/2}$ are invariant operator ideals in \mathcal{N} satisfying the “Cauchy-Schwarz” inequality: $\|XY\|_{\mathcal{I}} \leq \|X\|_{\mathcal{I}^{1/2}}\|Y\|_{\mathcal{I}^{1/2}}$, then*
(1) \mathcal{I}_{F_0} is a real vector space and if $F_1 \in F_0 + \mathcal{I}_{F_0}$ then

$$\mathcal{I}_{F_0} = \{Y \in \mathcal{I}_{sa}^{1/2} \mid 1 - (F_1 + Y)^2 \in \mathcal{I}\}$$

so that the definition is independent of the base point.

(2) In the norm $\|X\|_{F_0} = \|X\|_{\mathcal{I}^{1/2}} + \|XF_0 + F_0X\|_{\mathcal{I}}$, \mathcal{I}_{F_0} is a real Banach space and different base points define equivalent norms.

(3) If $\{F_n\}$ and F are in $F_0 + \mathcal{I}_{F_0}$ then $\|F_n - F\| \rightarrow 0$ if and only if $\|F_n - F\|_{\mathcal{I}^{1/2}} \rightarrow 0$ and $\|(1 - F_n^2) - (1 - F^2)\|_{\mathcal{I}} \rightarrow 0$.

Proof. The first part of the lemma is immediate from the definition and note (1) on page 678 of [CP1]. For the second part choose a Cauchy sequence $\{X_n\}$ in \mathcal{I}_{F_0} then there is a limit say X in $\mathcal{I}^{1/2}$ norm. As $\|\cdot\|_{\mathcal{I}^{1/2}}$ dominates the operator norm, $X_n \rightarrow X$ in \mathcal{N} . Similarly there is a limit Z of $F_0X_n + X_nF_0$ in \mathcal{I} norm and therefore in \mathcal{N} as well. Hence $Z = F_0X + XF_0$ is in \mathcal{I} so that $X_n \rightarrow X$ in the norm on \mathcal{I}_{F_0} . To see that the norm is independent of the base point let $F_1 = F_0 + Y$ and observe that

$$\begin{aligned} \|X\|_{F_1} &= \|X\|_{\mathcal{I}^{1/2}} + \|XF_1 + F_1X\|_{\mathcal{I}} \\ &\leq \|X\|_{\mathcal{I}^{1/2}} + \|XF_0 + F_0X\|_{\mathcal{I}} + \|XY + YX\|_{\mathcal{I}} \\ &\leq \|X\|_{F_0} + 2\|XY\|_{\mathcal{I}} \\ &\leq \|X\|_{F_0} + 2\|X\|_{\mathcal{I}^{1/2}}\|Y\|_{\mathcal{I}^{1/2}} \\ &\leq \|X\|_{F_0}(1 + 2\|Y\|_{\mathcal{I}^{1/2}}). \end{aligned}$$

The reverse inequality is similar.

The third part of the lemma is immediate. \square

Lemma B.13. *With \mathcal{I} as in the previous lemma, the map from $\mathcal{I}_{F_0} \rightarrow \mathcal{I}$ given by $X \mapsto 1 - (F_0 + X)^2$ is continuous.*

Proof. We have

$$\begin{aligned}
& \|1 - (F_0 + X)^2 - (1 - (F_0 + Y)^2)\|_{\mathcal{I}} \\
& \leq \|Y^2 - X^2\|_{\mathcal{I}} + \|(Y - X)F_0 + F_0(Y - X)\|_{\mathcal{I}} \\
& \leq \frac{1}{2} \|(Y - X)(Y + X) + (Y + X)(Y - X)\|_{\mathcal{I}} + \|Y - X\|_{F_0} \\
& \leq \|Y - X\|_{\mathcal{I}^{1/2}} \|Y + X\|_{\mathcal{I}^{1/2}} + \|Y - X\|_{F_0} \\
& \leq \|Y - X\|_{F_0} \|Y + X\|_{F_0} + \|Y - X\|_{F_0}.
\end{aligned}$$

□

Corollary B.14. *For a self-adjoint $F_0 \in \mathcal{N}$ with $1 - F_0^2 \in Li_0^q$ and $r \geq 0$ the map $F \mapsto |1 - F^2|^{-r} e^{-|1 - F^2|^{-1/q}}$ from the affine space $F_0 + (Li_0^q)_{F_0}$ to the trace-class operators is continuous.*

Proof. This follows from Lemma B.13 and Corollary B.10. □

Lemma B.15. *Suppose that \mathcal{I} and $\mathcal{I}^{1/2}$ are invariant operator ideals in \mathcal{N} satisfying the “Cauchy-Schwarz” inequality and let $F_0 \in \mathcal{N}_{sa}$ satisfy $1 - F_0^2 \in \mathcal{I}$. If $t \mapsto F_t \in F_0 + \mathcal{I}_{F_0}$ is a path, then it is C^1 in that space if and only if:*

- (1) $t \mapsto F_t$ is C^1 in $\mathcal{I}^{1/2}$ -norm, and
- (2) $t \mapsto (1 - F_t^2)$ is C^1 in \mathcal{I} -norm.

Proof. Suppose conditions (1) and (2) are satisfied, and suppose $F_t = F_0 + X_t$ so that $t \mapsto X_t$ is C^1 in $\mathcal{I}^{1/2}$ -norm. Then,

$$\mathcal{I}^{1/2} - \lim_{s \rightarrow t} \frac{1}{s - t} (X_s - X_t) = X'_t \text{ exists}$$

and $t \mapsto X'_t$ is $\mathcal{I}^{1/2}$ -norm continuous (thus, X'_t also exists in operator-norm and is operator-norm continuous).

Now by the product rule and the “Cauchy-Schwarz” inequality, we have that $t \mapsto X_t^2 \in \mathcal{I}$ is C^1 in the norm of \mathcal{I} . Hence,

$$t \mapsto (F_0 X_t + X_t F_0) = (1 - F_0^2 - X_t^2) - (1 - F_t^2)$$

is C^1 in the norm of \mathcal{I} by condition (2). Then:

$$\begin{aligned}
Z_t : &= \|\cdot\|_{\mathcal{I}} \cdot \frac{d}{dt} (F_0 X_t + X_t F_0) \\
&= \|\cdot\|_{\mathcal{I}} \cdot \frac{d}{dt} (F_0 X_t + X_t F_0) \\
&= \cdots = F_0 X'_t + X'_t F_0.
\end{aligned}$$

That is, the difference quotients for X'_t also converge to X'_t in the norm of \mathcal{I}_{F_0} .

The proof of the other implication is similar and a little easier. Since we do not use this implication anywhere, we omit the proof. □

APPENDIX C. INTEGRAL FORMULAE

In order to explicitly compute the derivatives used in Section 5 we need to be able to express the map $T \mapsto |T|^{-r} e^{-|T|^{-1}}$ in terms of the resolvents of T^2 , where T is self-adjoint and bounded and $r \geq 0$ (the cases $r = 0$ and $r = 3/2$ are the ones of interest). In order to do this, we are forced to consider Cauchy integrals along unbounded contours. In what follows, we take λ^r to be the principal branch of the usual analytic function of λ on $\mathbb{C} \setminus (-\infty, 0]$.

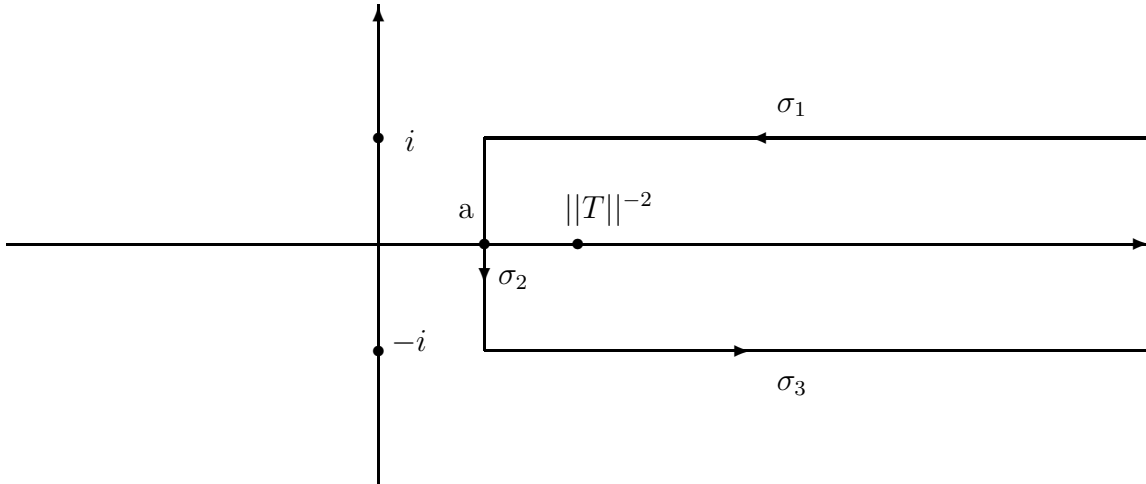
Lemma C.1. *For S bounded and self-adjoint and with $\lambda = \pm(t \pm i)$,*

$$\|(\lambda S - 1)^{-1}\| \leq (1 + t^2)^{1/2} \leq 1 + |t| \quad (C.1)$$

Proof. It suffices, by the functional calculus, to prove the numerical inequality $|(\lambda x - 1)^{-1}| \leq (1 + t^2)^{1/2}$ for all $x \in \mathbb{R}$ which follows by elementary calculus. \square

Lemma C.2. *For T bounded and self-adjoint and for any real a in $(0, \|T\|^{-2})$ we let $\sigma = \sigma_1 + \sigma_2 + \sigma_3$ be the piecewise smooth curve in \mathbb{C} with $\sigma_1(t) = -t + i$, $t \in (-\infty, -a]$, $\sigma_2(t) = a - ti$, $t \in [-1, -1]$ and $\sigma_3(t) = t - i$, $t \in [a, \infty)$. Then for $c \geq 0$, $k > 0$, $b > 0$ the integral $\int_{\sigma} T^2(\lambda T^2 - 1)^{-1} \lambda^c e^{-b\lambda^k} d\lambda$ converges absolutely in operator norm.*

Proof. The contour σ is pictured below.



Now, the integrand is a well-defined continuous function of λ as $\frac{1}{a} > \|T^2\|$ so that $\frac{1}{a} \notin \text{sp}(T^2)$. For $\lambda = \sigma_1(t) = -t + i$ (C.1) gives

$$\begin{aligned} \|T^2(\lambda T^2 - 1)^{-1} \lambda^c e^{-b\lambda^k}\| &\leq \|T\|^2 (1 + t^2)^{1/2} |\lambda|^c e^{-\Re(b\lambda^k)} \\ &\leq \|T\|^2 (1 + t^2)^{(c+1)/2} e^{-b(1+t^2)^{k/2}/2} \end{aligned}$$

as soon as $|k \arg \lambda| \leq \pi/3$. This is clearly integrable as $t \rightarrow -\infty$. The rest is similar. \square

Lemma C.3. *Let t_0 in $[0, \|T\|^2]$ be fixed and let σ be as above. Then*

$$\frac{1}{2\pi i} \int_{\sigma} t_0(\lambda t_0 - 1)^{-1} \lambda^c e^{-b\lambda^k} d\lambda$$

converges absolutely and equals $t_0^{-c}e^{-b/t_0^k}$ for $t_0 > 0$ and is zero for $t_0 = 0$.

Proof. We take a cutoff version, σ_N , of σ by truncating σ_1 and σ_3 to $\sigma'_1(t) = -t + i$, $t \in (-N, -a]$ and $\sigma'_3(t) = t - i$, $t \in [a, N)$ and joining the endpoints. Then for $\frac{1}{t_0} < N$ we get the value of the integral around σ_N as stated using Cauchy's theorem. It suffices to show that the integral along the complementary path, $\sigma'_N = \sigma - \sigma_N$, converges to 0 as $N \rightarrow \infty$. The integrals along the horizontal pieces go to zero as $N \rightarrow \infty$ using the previous lemma. The vertical piece where $\lambda(t) = N - ti$ for $t \in [-1, 1]$ can be estimated (when $N \geq 2/t_0$) using $\frac{t_0}{|\lambda t_0 - 1|} \leq t_0$. This bounds the vertical piece by $\frac{t_0}{\pi}(N^2 + 1)^{c/2}e^{-bN^k/2}$ provided $|k \arg(N \pm i)| \leq \pi/3$. Hence this piece also goes to zero as $N \rightarrow \infty$. \square

Lemma C.4. Let T be bounded and self-adjoint and let σ be as in the previous lemmas. Let $0 < a < \|T\|^{-2}$ then we have for $c \geq 0$ and $k > 0$ that the integral

$$\frac{1}{2\pi i} \int_{\sigma} T^2(\lambda T^2 - 1)^{-1} \lambda^c e^{-b\lambda^k} d\lambda$$

converges absolutely in operator norm to $(T^2)^{-c}e^{-b|T|^{-2k}}$ (which equals $|T|^{-2c}e^{-b|T|^{-2k}}$).

Proof. First let $\{E_x\}$ be the spectral resolution of T^2 (assumed as usual to be strong operator continuous from the right except at zero where we take $E_0 = 0$ while $\lim_{x \rightarrow 0^+} E_x$ is the kernel projection) and let $\xi \in H$. By the Spectral Theorem:

$$\begin{aligned} \langle (T^2)^{-c}e^{-b|T|^{-2k}} \xi, \xi \rangle &= \int_0^{\|T\|^2} x^{-c} e^{-b/x^k} d\langle E_x \xi, \xi \rangle \\ &= \int_0^{\|T\|^2} \left\{ \frac{1}{2\pi i} \int_{\sigma} x(\lambda x - 1)^{-1} \lambda^c e^{-b\lambda^k} d\lambda \right\} d\langle E_x \xi, \xi \rangle \\ &= \frac{1}{2\pi i} \int_{\sigma} \left\{ \int_0^{\|T\|^2} x(\lambda x - 1)^{-1} d\langle E_x \xi, \xi \rangle \right\} \lambda^c e^{-b\lambda^k} d\lambda \\ &= \frac{1}{2\pi i} \int_{\sigma} \langle T^2(\lambda T^2 - 1)^{-1} \xi, \xi \rangle \lambda^c e^{-b\lambda^k} d\lambda \\ &= \left\langle \left(\frac{1}{2\pi i} \int_{\sigma} T^2(\lambda T^2 - 1)^{-1} \lambda^c e^{-b\lambda^k} d\lambda \right) \xi, \xi \right\rangle \end{aligned}$$

as the last integral converges absolutely in operator norm. Observe that the change in the order of integration is justified for the following reasons. The measures $|d\lambda|$ and $d\langle E_x \xi, \xi \rangle$ are positive and σ -finite and the function $|x(\lambda x - 1)^{-1} \lambda^c e^{-b\lambda^k}|$ is continuous and hence product measurable. Moreover the iterated integral

$$\int_0^{\|T\|^2} \int_{\sigma} |x(\lambda x - 1)^{-1} \lambda^c e^{-b\lambda^k}| \cdot |d\lambda| d\langle E_x \xi, \xi \rangle$$

is clearly finite so an application of the theorems of Tonelli and Fubini justifies the interchange. \square

We now use the notation and results of Appendix B and the integral formula of the previous lemma to compute the exterior derivatives of the one-forms of Section 5.

Remarks. Note that if $g : [0, 1] \rightarrow L^1$ is continuous in the trace-norm and differentiable in the operator norm then $f = g^2$ is differentiable in *trace-norm* with $f'(s) = g'(s)g(s) + g(s)g'(s)$ and so $\frac{d}{ds}\tau(f(s)) = 2\tau(g(s)g'(s))$. The proof just uses the usual product rule method.

As usual F_0 is self-adjoint Breuer-Fredholm operator with $1 - F_0^2 \in Li_0^q$. Let $X \in (Li_0^q)_{F_0}$ and let $F_s = F + sX$ so that $F_s^* = F_s$ and hence $1 - F_s^2$ is in Li_0^q . Let $T_s = 1 - F_s^2$ and $T_0 = 1 - F_0^2$. To differentiate $s \mapsto |T_s|^{-r}e^{-|T_s|^{-1/q}}$ we factorise it as the square of $s \mapsto |T_s|^{-r/2}e^{-(1/2)|T_s|^{-1/q}}$ and apply the above remark noting that $s \mapsto T_s$ is Li^q -norm continuous so that $s \mapsto |T_s|^{-r/2}e^{-(1/2)|T_s|^{-1/q}}$ is continuous in trace-norm.

Theorem C.5. Assume the notation above and fix $r \geq 0$ with

$$g_r(T) = |T|^{-r/2}e^{-(1/2)|T|^{-1/q}}$$

for $T \in Li_0^q$. Then in trace norm

$$\frac{d}{ds}\big|_{s=0}(|1 - F_s^2|^{-r}e^{-|1 - F_s^2|^{-1/q}}) = \frac{d}{ds}\big|_{s=0}[g_r(T_s)]^2$$

exists and equals

$$\frac{i}{2\pi} \int_{\sigma} [g_r(T_0), R_{\lambda}(T_0) [[F_0, X]_+, T_0]_+ R_{\lambda}(T_0)]_+ m(\lambda) d\lambda$$

where $T_0 = 1 - F_0^2$; $[\cdot, \cdot]_+$ denotes the anticommutator; $R_{\lambda}(T) = (\lambda T^2 - 1)^{-1}$; and $m(\lambda) = \lambda^{r/4}e^{-(\lambda^{1/2q})/2}$.

Proof. By the remark it suffices to show that in operator norm $\frac{d}{ds}\big|_{s=0}(g_r(T_s))$ exists and equals

$$\frac{i}{2\pi} \int_{\sigma} R_{\lambda}(T_0) \{ (F_0 X + X F_0) T_0 + T_0 (F_0 X + X F_0) \} R_{\lambda}(T_0) m(\lambda) d\lambda$$

where $T_s = 1 - (F_0 + sX)^2$. Now,

$$\begin{aligned} & \frac{1}{s}(g_r(T_s) - g_r(T_0)) \\ &= \frac{1}{s2\pi i} \int_{\sigma} [T_s^2 R_{\lambda}(T_s) - T_0^2 R_{\lambda}(T_0)] m(\lambda) d\lambda \\ &= \frac{1}{s2\pi i} \int_{\sigma} (R_{\lambda}(T_s) [T_s^2 (\lambda T_0^2 - 1) - (\lambda T_s^2 - 1) T_0^2] R_{\lambda}(T_0) m(\lambda) d\lambda \\ &= \frac{1}{s2\pi i} \int_{\sigma} R_{\lambda}(T_s) (T_0^2 - T_s^2) R_{\lambda}(T_0) m(\lambda) d\lambda \\ &= \frac{1}{s2\pi i} \int_{\sigma} R_{\lambda}(T_s) \frac{1}{2} [(T_0 - T_s)(T_0 + T_s) + (T_0 + T_s)(T_0 - T_s)] R_{\lambda}(T_0) m(\lambda) d\lambda \\ &= \frac{-1}{4\pi i} \int_{\sigma} R_{\lambda}(T_s) [(F_0 X + X F_0 + sX^2), (T_0 + T_s)]_+ R_{\lambda}(T_0) m(\lambda) d\lambda \end{aligned} \tag{C.2}$$

where again $[\cdot, \cdot]_+$ denotes the anti-commutator. We need to check that we get convergence to the formal limit as $s \rightarrow 0$ of (C.2):

$$\frac{i}{2\pi} \int_{\sigma} R_{\lambda}(T_0)[(F_0X + XF_0)T_0 + T_0(F_0X + XF_0)]R_{\lambda}(T_0)m(\lambda)d\lambda. \quad (C.3)$$

Now let $h_s(\lambda)$ denote the continuous operator-valued function of λ which is the integrand of (C.2) and let $h(\lambda)$ denote the integrand of (C.3). It suffices to show that $h_s \rightarrow h$ uniformly on compact subsets (of σ) and that $\|h_s(\lambda)\| \leq k(\lambda)$ where $\int_{\sigma} k(\lambda)|d\lambda| < \infty$. For this to hold it suffices to show that

$$\|(\lambda T_s^2 - 1)^{-1} - (\lambda T_0^2 - 1)^{-1}\| \rightarrow 0$$

uniformly on compacta as the other factors converge uniformly on all of σ . Now, by the resolvent equation and Lemma C.1,

$$\begin{aligned} \|(\lambda T_s^2 - 1)^{-1} - (\lambda T_0^2 - 1)^{-1}\| &= \|(\lambda T_s^2 - 1)^{-1} \lambda (T_0^2 - T_s^2) (\lambda T_0^2 - 1)^{-1}\| \\ &\leq |\lambda|^3 \|T_0^2 - T_s^2\| \rightarrow 0 \end{aligned}$$

uniformly for λ in a bounded set. Let

$$C = \sup_{s \in [-1, 1]} \frac{1}{2} \|[(F_0X + XF_0 + sX^2)(T_0 + T_s) + (T_0 + T_s)(F_0X + XF_0 + sX^2)]\|.$$

Then

$$\|h_s(\lambda)\| \leq C|\lambda|^{2+r/4} e^{-\Re(\lambda^{1/2q})/2}$$

which is integrable on σ . □

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